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MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CENTRE NUMBER S

INDEX NUMBER

* 5 1 8 4 7 9 7 5 3 5 *

MATHEMATICS

9758/01

Paper 1

October/November 2020

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **22** printed pages and **2** blank pages.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

JC20 11_9758_01/5R

[Turn over

1 A plane π_1 contains two vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$.

(i) Find a vector normal to π_1. [2]

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$$

A plane π_2 has equation $4x + 5y - 6z = 0$.

(ii) Find the acute angle between π_1 and π_2. [2]

$$n_1 = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}$$

$$\theta = \cos^{-1} \left| \frac{\begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}}{\sqrt{44}\sqrt{77}} \right|$$

$$= \cos^{-1} \left| \frac{38}{\sqrt{44}\sqrt{77}} \right|$$

$$= 49.2^\circ$$

2 A curve has equation $\frac{x^2}{1+x^2} + \frac{y^2}{1+y^2} = x^3y^5$. Find the equation of the tangent to the curve at the point $(1, 1)$. Give your answer in the form $ax + by = c$, where a, b and c are integers. [6]

Given $\frac{x^2}{1+x^2} + \frac{y^2}{1+y^2} = x^3y^5$

$$x^2 + \frac{y^2(1+x^2)}{1+y^2} = x^3y^5(1+x^2)$$

$$x^2(1+y^2) + y^2(1+x^2) = x^3y^5(1+x^2)(1+y^2)$$

$$x^2 + y^2 + 2x^2y^2 = x^3y^5(1+x^2+y^2+x^2y^2)$$

Implicit differentiation w.r.t x :

$$2x + 2y\frac{dy}{dx} + 2x^2\left(2y\frac{dy}{dx}\right) + y^2(4x) = (x^3y^5)\cdot\left[2x + 2y\frac{dy}{dx} + x^2\cdot 2y\frac{dy}{dx} + y^2(2x)\right] + (1+x^2+y^2+x^2y^2)\cdot\left[x^3\cdot 5y^4\frac{dy}{dx} + y^5\cdot 3x^2\right]$$

At $(1,1)$,

$$2 + 2\frac{dy}{dx} + 4\frac{dy}{dx} + 4 = (1)\cdot\left[2 + 2\frac{dy}{dx} + 2\frac{dy}{dx} + 2\right] + (4)\cdot\left[5\frac{dy}{dx} + 3\right]$$

$$6 + 6\frac{dy}{dx} = \left(4 + 4\frac{dy}{dx}\right) + \left(20\frac{dy}{dx} + 12\right)$$

$$18\frac{dy}{dx} = -10$$

$$\frac{dy}{dx} = -\frac{5}{9}$$

$\therefore$ Equation of tangent:

$$y - 1 = -\frac{5}{9}(x-1)$$

$$y = -\frac{5}{9}x + \frac{14}{9}$$

$$9y = -5x + 14$$

$$5x + 9y = 14$$

hence, $a = 5$, $b = 9$, $c = 14$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

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3 It is given that $f(x) = \ln(1 + \sin 3x)$.

(i) Show that $f''(x) = \dfrac{k}{1 + \sin 3x}$, where k is a constant to be found. [3]

$$f(x) = \ln(1 + \sin 3x)$$

$$f'(x) = \frac{1}{1 + \sin 3x} \cdot 3\cos 3x$$

$$= \frac{3\cos 3x}{1 + \sin 3x}$$

$$f''(x) = \frac{(1 + \sin 3x)(-9\sin 3x) - (3\cos 3x)(3\cos 3x)}{(1 + \sin 3x)^2}$$

$$= \frac{-9\sin 3x - 9\sin^2 3x - 9\cos^2 3x}{(1 + \sin 3x)^2}$$

$$= \frac{-9(\sin 3x + \sin^2 3x + \cos^2 3x)}{(1 + \sin 3x)^2}$$

$$= \frac{-9(1 + \sin 3x)}{(1 + \sin 3x)^2}$$

$$= \frac{-9}{1 + \sin 3x}$$

hence, $k = -9$

(ii) Hence find the first three non-zero terms of the Maclaurin expansion of $f(x)$. [4]

When $x=0$, $f(0) = \ln(1+0) = 0$

$f'(0) = \frac{3}{1} = 3$

$f''(0) = \frac{-9}{1} = -9$

Since $f''(x) = -9(1+\sin 3x)^{-1}$

$f^3(x) = 9(1+\sin 3x)^{-2}(3\cos 3x)$

$= \frac{27\cos 3x}{(1+\sin 3x)^2}$

$\therefore f^3(0) = \frac{27}{1} = 27$

Hence, $f(x) = 3x + \frac{x^2}{2}(-9) + \frac{x^3}{6}(27) + \cdots$

$= \underline{3x - \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

4 Do not use a calculator in answering this question.

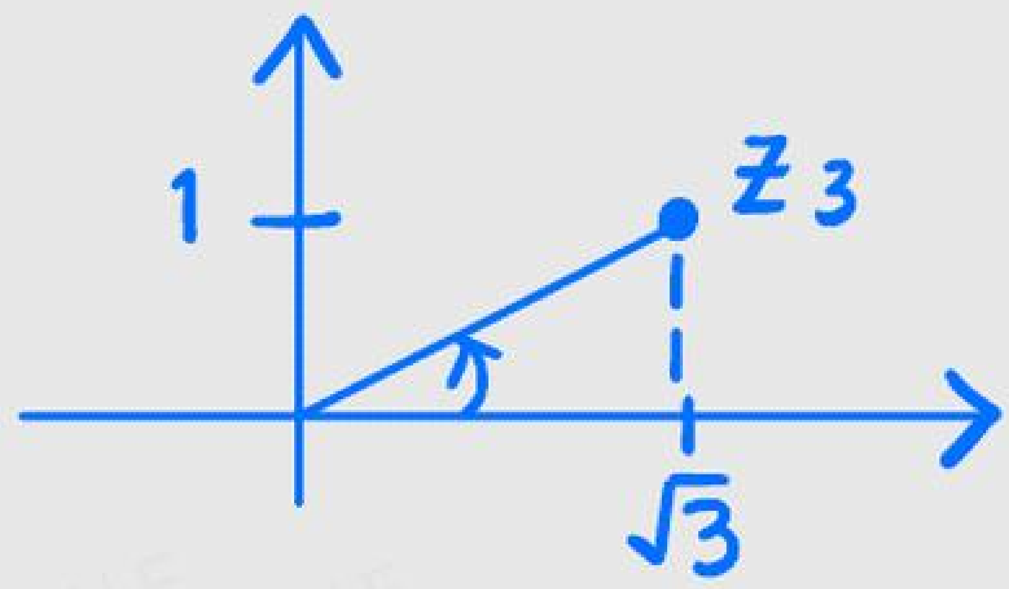
Three complex numbers are $z_1 = 1 + \sqrt{3}\mathrm{i}$, $z_2 = 1 - \mathrm{i}$ and $z_3 = 2(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi)$.

(i) Find $\dfrac{z_1}{z_2 z_3}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [4]

$z_3 = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

$|z_3| = \sqrt{3+1} = 2$

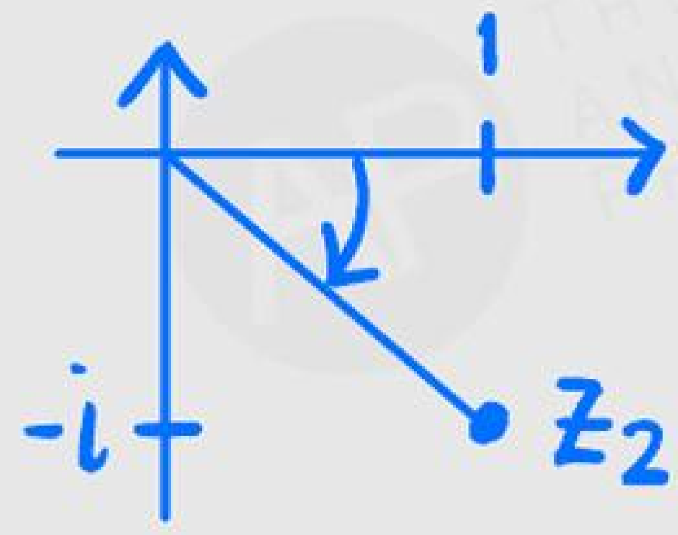
$\arg z_3 = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$



$z_2 = 1 - i$

$|z_2| = \sqrt{1+1} = \sqrt{2}$

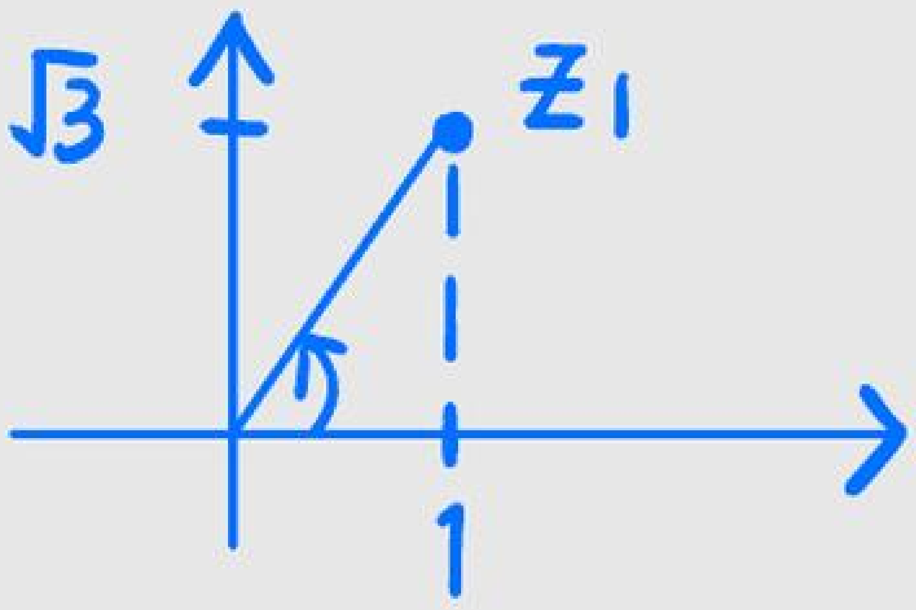
$\arg z_2 = -\frac{\pi}{4}$



$z_1 = 1 + \sqrt{3}i$

$|z_1| = \sqrt{1+3} = 2$

$\arg z_1 = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$



$$\left|\frac{z_1}{z_2 z_3}\right| = \frac{2}{\sqrt{2}\cdot 2} = \frac{1}{\sqrt{2}}$$

$$\arg\left(\frac{z_1}{z_2 z_3}\right) = \arg z_1 - (\arg z_2 + \arg z_3)$$

$$= \frac{\pi}{3} + \frac{\pi}{4} - \frac{\pi}{6} = \frac{5\pi}{12}$$

$$\therefore \frac{z_1}{z_2 z_3} = \frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

A fourth complex number, z_4, is such that $\frac{z_1 z_4}{z_2 z_3}$ is purely imaginary and $\left|\frac{z_1 z_4}{z_2 z_3}\right| = 1$.

(ii) Find the possible values of z_4 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

Since $\frac{z_1}{z_2 z_3} = \frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

$\because \left|\frac{z_1}{z_2 z_3}\right| = \frac{1}{\sqrt{2}}$ and $\left|\frac{z_1 z_4}{z_2 z_3}\right| = 1$

$\therefore \underline{|z_4| = \sqrt{2}}$

since $\frac{z_1 z_4}{z_2 z_3}$ is purely imaginary,

$\arg\left(\frac{z_1 z_4}{z_2 z_3}\right) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

If $\arg\left(\frac{z_1}{z_2 z_3}\right) z_4 = \frac{\pi}{2}$

$\arg\left(\frac{z_1}{z_2 z_3}\right) + \arg z_4 = \frac{\pi}{2}$

$\therefore \arg z_4 = \frac{\pi}{2} - \frac{5\pi}{12}$

$= \underline{\frac{\pi}{12}}$

$z_4 = \underline{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}$

If $\arg\left(\frac{z_1}{z_2 z_3}\right) z_4 = -\frac{\pi}{2}$

$\arg\left(\frac{z_1}{z_2 z_3}\right) + \arg z_4 = -\frac{\pi}{2}$

$\therefore \arg z_4 = -\frac{\pi}{2} - \frac{5\pi}{12}$

$= \underline{-\frac{11\pi}{12}}$

$z_4 = \underline{\sqrt{2}\left(\cos -\frac{11\pi}{12} + i\sin -\frac{11\pi}{12}\right)}$

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5 (a) Given that **a** and **b** are non-zero vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$, find the relationship between **a** and **b**. [2]

$\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{b} \times \underset{\sim}{a}$

$(\underset{\sim}{a} \times \underset{\sim}{b}) - (\underset{\sim}{b} \times \underset{\sim}{a}) = 0$

$(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{a} \times \underset{\sim}{b}) = 0$

$2(\underset{\sim}{a} \times \underset{\sim}{b}) = 0$

$\therefore \underset{\sim}{a} \times \underset{\sim}{b} = 0$

Hence $\underset{\sim}{a} \parallel \underset{\sim}{b}$, i.e. $\underset{\sim}{a} = k\underset{\sim}{b}, \ k \in \mathbb{R}$

(b) The points P, Q and R have position vectors **p**, **q** and **r** respectively. The points P and Q are fixed and R varies.

(i) Given that **q** is non-zero and $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$, describe geometrically the set of all possible positions of the point R. [3]

$(\underset{\sim}{r} - \underset{\sim}{p}) \times \underset{\sim}{q} = 0$

$\therefore (\underset{\sim}{r} - \underset{\sim}{p}) = \lambda \underset{\sim}{q}, \ \lambda \in \mathbb{R}$

$\underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{q}, \ \lambda \in \mathbb{R}$

Since R varies, $\underset{\sim}{r}$ represents the position vector of a variable point R that lies on the vector line parallel to $\underset{\sim}{q}$ and which passes through point P.

(ii) Given instead that $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$ and that $(\mathbf{r} - \mathbf{p}).\mathbf{q} = 0$, find the relationship between x, y and z. Describe the set of all possible positions of the point R in this case. [4]

$$(\underset{\sim}{r} - \underset{\sim}{p}) \cdot \underset{\sim}{q} = 0$$

$$\underset{\sim}{r} \cdot \underset{\sim}{q} = \underset{\sim}{p} \cdot \underset{\sim}{q}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

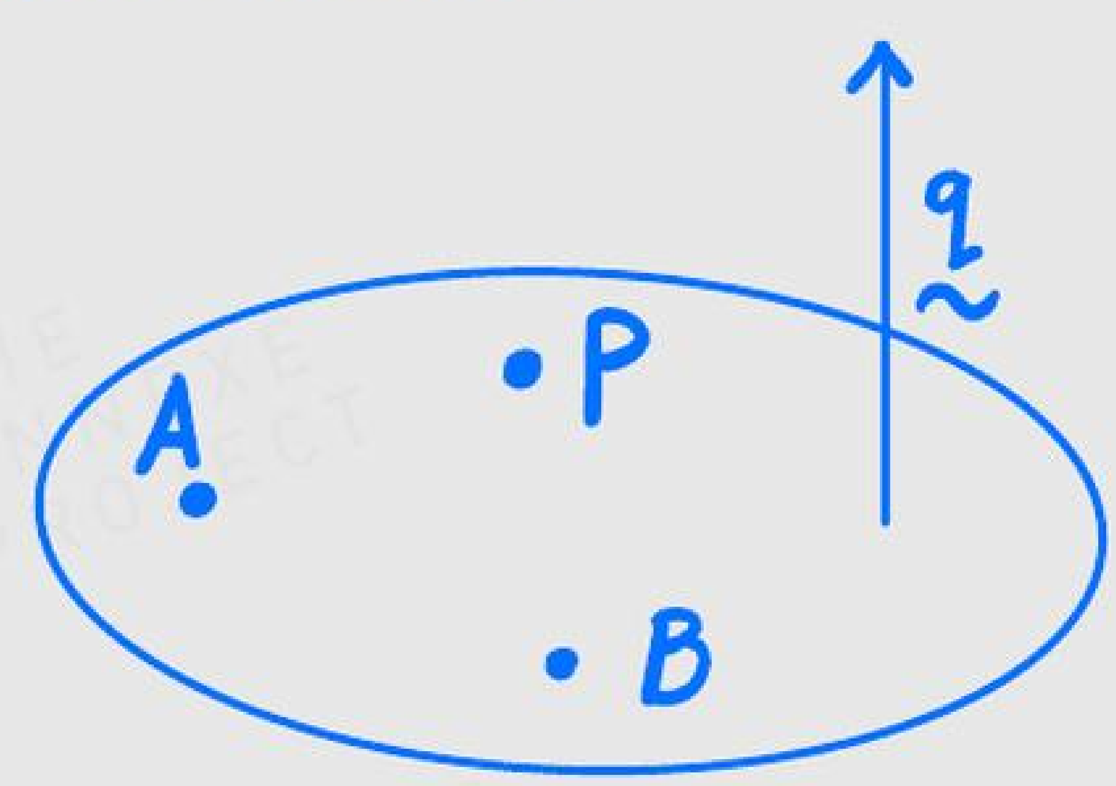
$$= -3 - 10 + 8$$

$$= -5$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = -5$$

$$3x - 5y + 2z = -5$$

$\overrightarrow{OR}$ represents all the possible position vectors that lie on the vector plane with normal vector $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$.



Let A and B be 2 points on the plane,

and $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

then, $\overrightarrow{PA} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$

$\overrightarrow{PB} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -5 \end{pmatrix}$

$\overrightarrow{OR} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -2 \\ -5 \end{pmatrix}$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$

i.e. $\overrightarrow{OR} = \begin{pmatrix} -1 + \lambda \\ 2 - \lambda - 2\mu \\ 4 - 4\lambda - 5\mu \end{pmatrix}$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$

6 The complex number z satisfies the equation

$$z^2(2+\mathrm{i}) - 8\mathrm{i}z + t = 0,$$

where t is a real number. It is given that one root is of the form $k + k\mathrm{i}$, where k is real and non-zero.

Find t and k, and the other root of the equation. [8]

$$(k+ki)^2(2+i) - 8i(k+ki) + t = 0$$

$$(\cancel{k^2} + 2k^2i - \cancel{k^2})(2+i) - 8ki + 8k + t = 0$$

$$4k^2i - 2k^2 - 8ki + 8k + t = 0$$

$$(-2k^2 + 8k + t) + i(4k^2 - 8k) = 0 + 0i$$

By comparison of coefficients:

$$4k^2 - 8k = 0$$

$$4k(k-2) = 0$$

$k = 0$ (rej.) or $k = 2$

$$-2k^2 + 8k + t = 0$$

$$-2(2^2) + 8(2) + t = 0$$

$$-8 + 16 + t = 0$$

$$t = -8$$

Hence, $z^2(2+i) - 8iz - 8 = 0$

Solving for z: $z = \dfrac{-(-8i) \pm \sqrt{(-8i)^2 - 4(2+i)(-8)}}{2(2+i)}$

$$= \frac{8i \pm \sqrt{-64 + 64 + 32i}}{4+2i}$$

$$= \frac{8i \pm (4+4i)}{4+2i}$$

$$= 2+2i \text{ or } -\tfrac{2}{5} + \tfrac{6}{5}i$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

6 **[Continued]**

7 **Do not use a calculator in answering this question.**

It is given that $f(x) = 2 - \sin 4x$.

(i) Find $\int f(x)\,dx$. [1]

$$\int 2 - \sin 4x \, dx = \underline{\boxed{2x + \frac{\cos 4x}{4} + C}}$$

(ii) Find the exact value, in terms of π, of $\int_0^{\frac{1}{2}\pi} x f(x)\,dx$. [4]

$$\int x \sin 4x \, dx = \frac{-x\cos 4x}{4} - \int \frac{-\cos 4x}{4}\,dx$$

Let $u = x$

$\frac{du}{dx} = 1$

Let $dv = \sin 4x$

$v = \frac{-\cos 4x}{4}$

$$= \frac{-x\cos 4x}{4} + \frac{1}{4}\int \cos 4x \, dx$$

$$= \frac{-x\cos 4x}{4} + \frac{\sin 4x}{16} + C$$

$$\therefore \int_0^{\frac{\pi}{2}} 2x - x\sin 4x \, dx$$

$$= \left[x^2 + \frac{x\cos 4x}{4} - \frac{\sin 4x}{16} \right]_0^{\frac{\pi}{2}}$$

$$= \underline{\boxed{\frac{\pi^2}{4} + \frac{\pi}{8}}}$$

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(iii) Find the exact value, in terms of π, of $\int_0^{\frac{1}{2}\pi} (\mathrm{f}(x))^2 \, \mathrm{d}x$. [4]

$$\int_0^{\frac{\pi}{2}} (2-\sin 4x)^2 \, dx = \int_0^{\frac{\pi}{2}} 4 - 4\sin 4x + \sin^2 4x \, dx$$

$$= \int_0^{\frac{\pi}{2}} 4 - 4\sin 4x + \tfrac{1}{2} - \tfrac{1}{2}\cos 8x \, dx$$

Double Angle Formulae

$\cos 8x = 1 - 2\sin^2 4x$

$\therefore 2\sin^2 4x = 1 - \cos 8x$

$\sin^2 4x = \frac{1}{2}(1 - \cos 8x)$

$$= \left[\tfrac{9}{2}x + \cos 4x - \tfrac{1}{16}\sin 8x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\tfrac{9\pi}{4} + 1\right] - [1]$$

$$= \frac{9\pi}{4}$$

8 **(a)** The 1st term of an arithmetic series is 4 and the 5th term is 10.

(i) Find the 30th term of this series. [2]

Given $a = 4$ and $a + 4d = 10$

Hence, $4 + 4d = 10$

$4d = 6$

$d = \frac{3}{2}$

$T_{30} = 4 + 29\left(\frac{3}{2}\right) = \frac{95}{2}$

(ii) Find the sum of the 21st term to the 50th term inclusive of this series. [3]

$S_{50} - S_{20} = \frac{50}{2}\left[2(4) + 49\left(\frac{3}{2}\right)\right] - \frac{20}{2}\left[2(4) + 19\left(\frac{3}{2}\right)\right]$

$= 1672.5$

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(b) The 1st term of a geometric series is 4 and the 5th term is 1.6384 where the common ratio is positive.

(i) Find the sum to infinity of this series. [2]

$a = 4, \quad ar^4 = 1.6384$

$\therefore \quad r^4 = \frac{1.6384}{4}$

$r = \pm 0.8$

since r is positive, $r = 0.8$

$\therefore S_\infty = \frac{4}{1-0.8} = \underline{20}$

(ii) Given that the sum of the first n terms is greater than 19.6, show that

$$0.8^n < 0.02.$$

Hence find the smallest possible value of n. [5]

$S_n > 19.6$

$\frac{a(1-r^n)}{1-r} > 19.6$

$(S_\infty)(1-r^n) > 19.6$

$1 - r^n > \frac{19.6}{20}$

$1 - 0.8^n > 0.98$

$0.8^n < 0.02$ (shown).

$n \ln 0.8 < \ln 0.02$

$n > 17.531$

$\therefore n_{\text{least}} = \underline{18}$

9 **(i)** By considering the gradients of two lines, explain why $\tan^{-1}(2) - \tan^{-1}(-\frac{1}{2}) = \frac{1}{2}\pi$. [1]

gradient = $\tan\theta$

i.e. $m = \tan\theta$

$\theta = \tan^{-1} m$

Hence, gradient of 1st line $m_1 = 2$

gradient of 2nd line $m_2 = -\frac{1}{2}$

$m_1 \cdot m_2 = -1$, hence both lines are perpendicular.

The curves C_1 and C_2 have equations $y = \frac{1}{x^2+1}$ and $y = \frac{k}{3x+4}$ respectively, where k is a constant and $k > 0$.

(ii) Find the set of values of k such that C_1 and C_2 intersect. [3]

Let $\frac{1}{x^2+1} = \frac{k}{3x+4}$

$kx^2 + k = 3x + 4$

$kx^2 - 3x + (k-4) = 0$

For C_1 and C_2 to intersect,

$b^2 - 4ac \geqslant 0$

$9 - 4k(k-4) \geqslant 0$

$-4k^2 + 16k + 9 \geqslant 0$

$4k^2 - 16k - 9 \leqslant 0$

$(2k+1)(2k-9) \leqslant 0$

$-\frac{1}{2} \leqslant k \leqslant \frac{9}{2}$

Since $k > 0$, then $0 < k \leqslant \frac{9}{2}$

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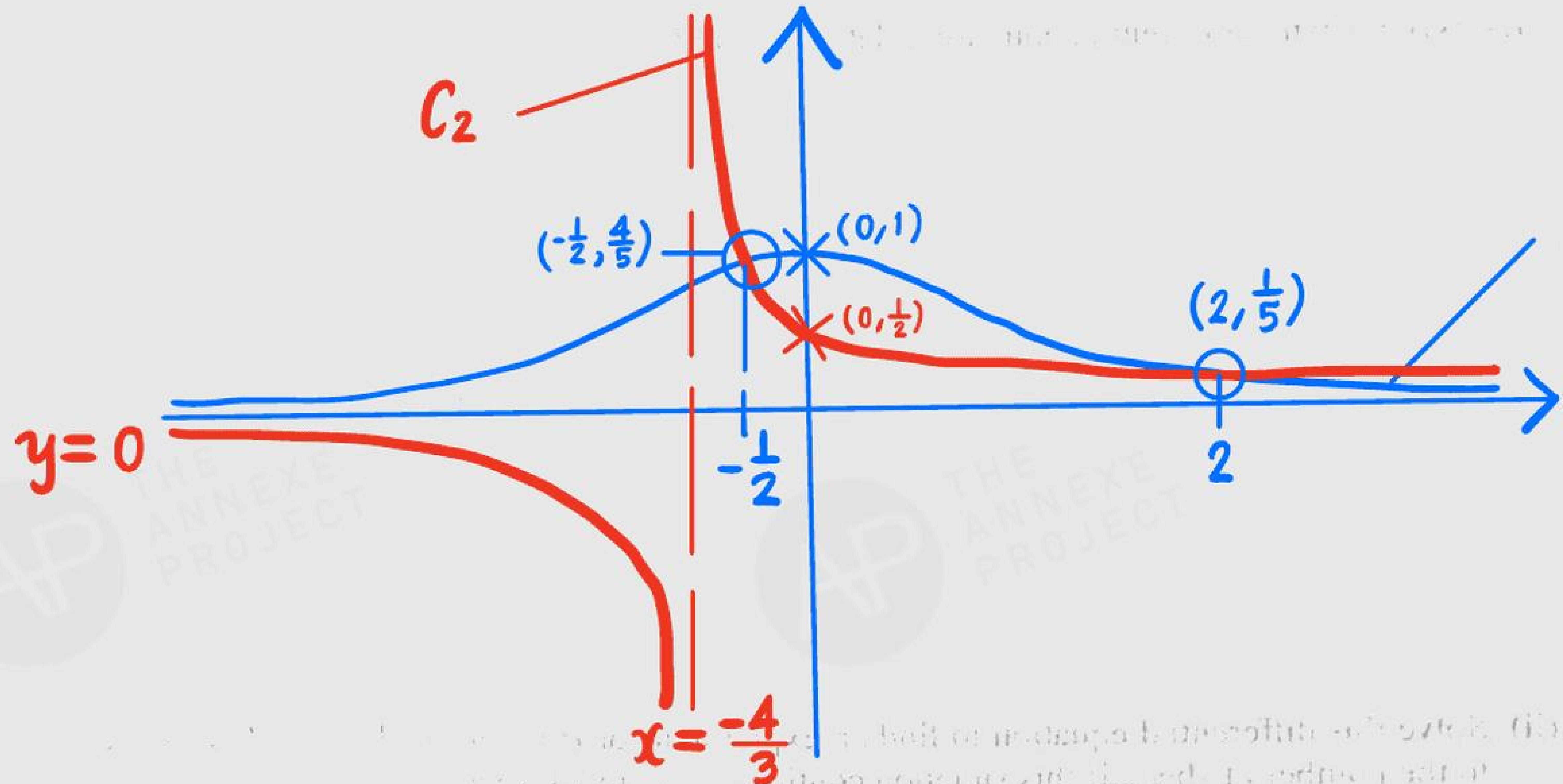
$y = \dfrac{1}{x^2+1}$ and $y = \dfrac{k}{3x+4}$

It is now given that $k = 2$.

(iii) Sketch C_1 and C_2 on the same graph, giving the coordinates of any points where C_1 or C_2 cross the axes and the equations of any asymptotes. [3]



(iv) Find the exact area of the region bounded by C_1 and C_2, simplifying your answer. [5]

$$\int_{-\frac{1}{2}}^{2} \frac{1}{x^2+1} - \frac{2}{3x+4} \, dx$$

$$= \left[\tan^{-1} x - \frac{2}{3}\ln(3x+4)\right]_{-\frac{1}{2}}^{2}$$

$$= \left[\tan^{-1} 2 - \frac{2}{3}\ln 10\right] - \left[\tan^{-1}\left(-\frac{1}{2}\right) - \frac{2}{3}\ln\frac{5}{2}\right]$$

$$= \left(\tan^{-1} 2 - \tan^{-1}\left(-\frac{1}{2}\right)\right) + \frac{2}{3}\left[\ln\frac{5}{2} - \ln 10\right]$$

$$= \frac{\pi}{2} + \frac{2}{3}\ln\frac{1}{4}$$

$$= \frac{\pi}{2} + \frac{2}{3}\left[\ln 1 - \ln 4\right]$$

$$= \frac{\pi}{2} - \frac{2}{3}\ln 4$$

$$= \frac{\pi}{2} - \frac{4}{3}\ln 2 \text{ sq. units.}$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

10 Scientists are investigating the effect of disease on the number of sheep on a small island. They discover that every year the death rate of the sheep is greater than the birth rate of the sheep. The difference every year between the death rate and the birth rate for the population of sheep on the island is 3%. The number of sheep on the island is P at a time t years after the scientists begin observations.

(i) Write down a differential equation relating P and t. [2]

Given birth rate − death rate $= -3\% \times P$

$= -0.03P$

$\frac{dP}{dt} =$ birth rate − death rate

$\therefore \frac{dP}{dt} = -0.03P$

(ii) Solve this differential equation to find an expression for P in terms of t. Explain what happens to the number of sheep if this situation continues over many years. [4]

$\int \frac{1}{P}\, dP = \int -0.03 \; dt$

$\ln P = -0.03t + C$

$P = e^{-0.03t + C}$

$P = e^{C} e^{-0.03t}$

$P = Ae^{-0.03t}$ where $A = e^{C}$.

As $t \to \infty$, $e^{-0.03t} \to 0$.

Hence, $P \longrightarrow 0$.

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

The scientists import sheep at a constant uniform rate of n sheep per year. (The difference every year between the death rate and the birth rate remains at 3%.)

(iii) Write down a differential equation to model the new situation. [2]

$$\frac{dP}{dt} = -0.03P + n$$

(iv) Solve the differential equation to find an expression for P in terms of t and n. [4]

$$\int \frac{1}{n-0.03P}\,dP = \int dt$$

$$\frac{1}{-0.03}\int \frac{-0.03}{n-0.03P}\,dP = \int dt$$

$$-\frac{100}{3}\ln|n-0.03P| = t + C$$

$$\ln|n-0.03P| = \frac{-3}{100}t - \frac{3C}{100}$$

$$n-0.03P = \pm e^{-\frac{3}{100}t - \frac{3C}{100}}$$

$$= Ae^{-\frac{3}{100}t} \quad \text{where } A = \pm e^{-\frac{3C}{100}}$$

$$0.03P = n - Ae^{-0.03t}$$

$$P = \frac{100}{3}\left(n - Ae^{-0.03t}\right)$$

(v) Given that the number of sheep settles down to 500 after many years, find n. [2]

As $t \to \infty$, $Ae^{-0.03t} \to 0$

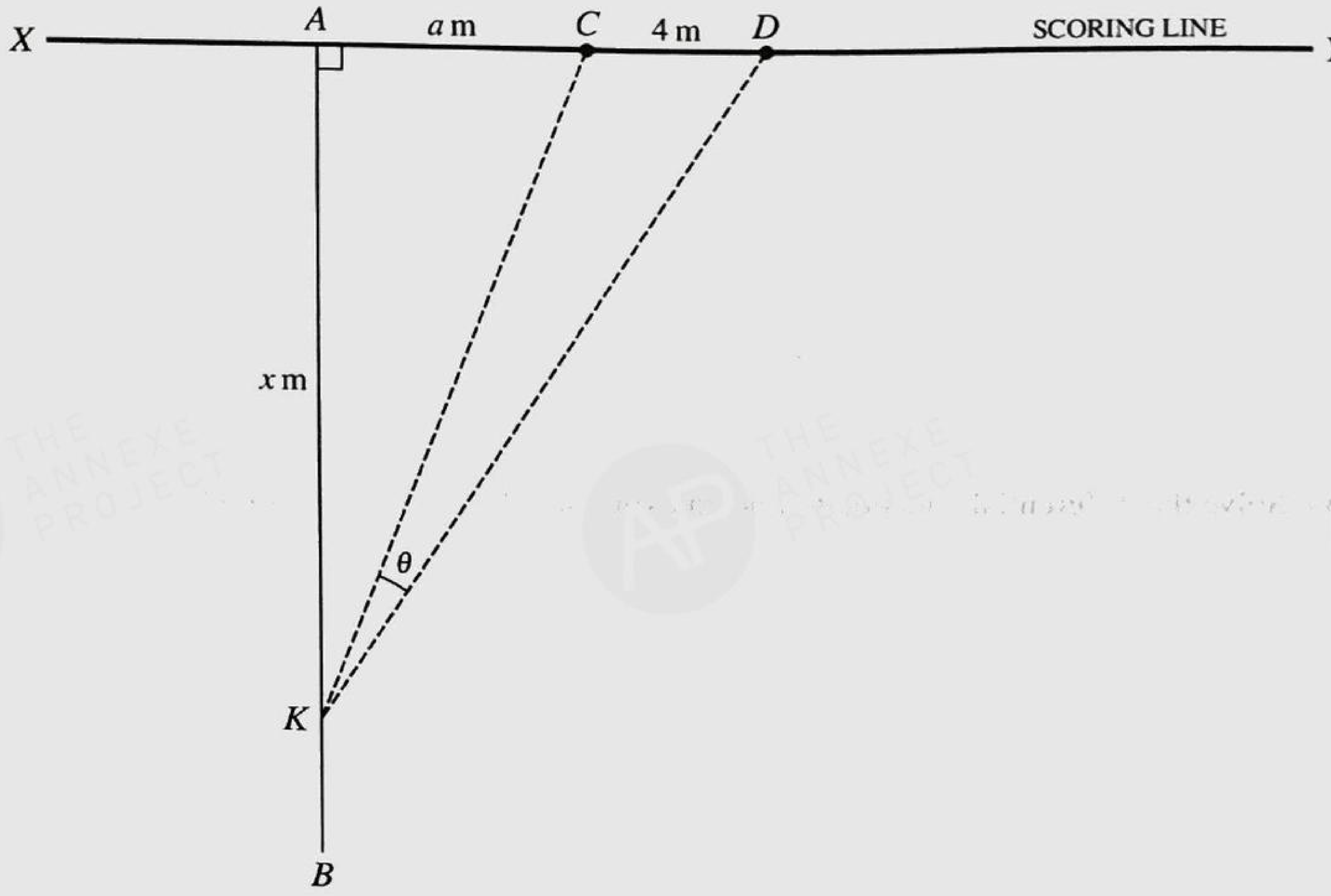
$\therefore P \longrightarrow \frac{100}{3}n$

Let $\frac{100}{3}n = 500$

$n = 15$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

11 In sport science, studies are made to optimise performance in all aspects of sport from fitness to technique.



In a game, a player scores 3 points by carrying the ball over the scoring line, shown in the diagram as XY. When a player has scored these 3 points, an extra point is scored if the ball is kicked between two fixed vertical posts at C and D. The kick can be taken from any point on the line AB, where A is the point at which the player crossed the scoring line and AB is perpendicular to XY.

The distance CD is 4 m; XC is equal to DY; the point A is a distance a m from C and A lies between X and C. The kick is taken from the point K, where AK is x m. The angle CKD is θ (see diagram).

(i) By expressing θ as the difference of two angles, or otherwise, show that

$$\tan\theta = \frac{4x}{x^2 + 4a + a^2}.$$ [3]

$\theta = \angle AKD - \angle AKC$

$$\tan\theta = \tan(\angle AKD - \angle AKC)$$

$$= \frac{\tan\angle AKD - \tan\angle AKC}{1 + \tan\angle AKD \tan\angle AKC}$$

$$= \frac{\frac{a+4}{x} - \frac{a}{x}}{1 + \left(\frac{a+4}{x}\right)\left(\frac{a}{x}\right)} = \frac{\frac{4}{x}}{1 + \frac{a(a+4)}{x^2}} = \frac{4x}{x^2 + a(a+4)}$$

$$= \frac{4x}{x^2 + 4a + a^2} \text{ (shown).}$$

(ii) Find, in terms of a, the value of x which maximises $\tan\theta$, simplifying your answer. Find also the corresponding value of $\tan\theta$. (You need not show that your answer gives a maximum.) [3]

Let T be $\tan\theta$: $\quad T = \dfrac{4x}{x^2 + 4a + a^2}$

$$\frac{dT}{dx} = \frac{(x^2 + 4a + a^2)(4) - 4x(2x)}{(x^2 + 4a + a^2)^2}$$

$$= \frac{4x^2 - 8x^2 + 4a^2 + 16a}{(x^2 + 4a + a^2)^2}$$

$$= \frac{-4x^2 + 4a^2 + 16a}{(x^2 + 4a + a^2)^2}$$

Let $\frac{dT}{dx} = 0$; i.e. $-4x^2 + 4a^2 + 16a = 0$

$$4x^2 = 4a^2 + 16a$$

$$x^2 = a^2 + 4a$$

$$x = \sqrt{a^2 + 4a}$$

$(\because x > 0)$

When $x = \sqrt{a^2 + 4a}$,

$$\tan\theta = \frac{4\sqrt{a^2+4a}}{a^2 + 4a + 4a + a^2} = \frac{4\sqrt{a^2+4a}}{2a^2 + 8a} = \frac{2\sqrt{a^2+4a}}{a(a+4)}$$

$$= \frac{2}{\sqrt{a(a+4)}}$$

11 [Continued]

The point corresponding to the value of x found in part **(ii)** is called the optimal point. The corresponding value of θ is called the optimal angle.

(iii) Explain why a player may decide not to take the kick from the optimal point. [1]

At the optimal point, the player is still of considerable distance from CD. Hence, he may choose to kick only when he moves closer towards A.

(iv) Show that, when θ is the optimal angle, $\tan KDA = \sqrt{\dfrac{a}{4+a}}$. Find the approximate value of angle KDA when a is much greater than 4. [3]

$$\tan \angle KDA = \frac{x}{a+4} = \frac{\sqrt{a^2+4a}}{a+4}$$
$$= \sqrt{\frac{a^2+4a}{(a+4)^2}}$$
$$= \sqrt{\frac{a(a+4)}{(a+4)^2}}$$
$$= \sqrt{\frac{a}{a+4}} \text{ (shown).}$$

When a is large, $\tan \angle KDA \approx 1$

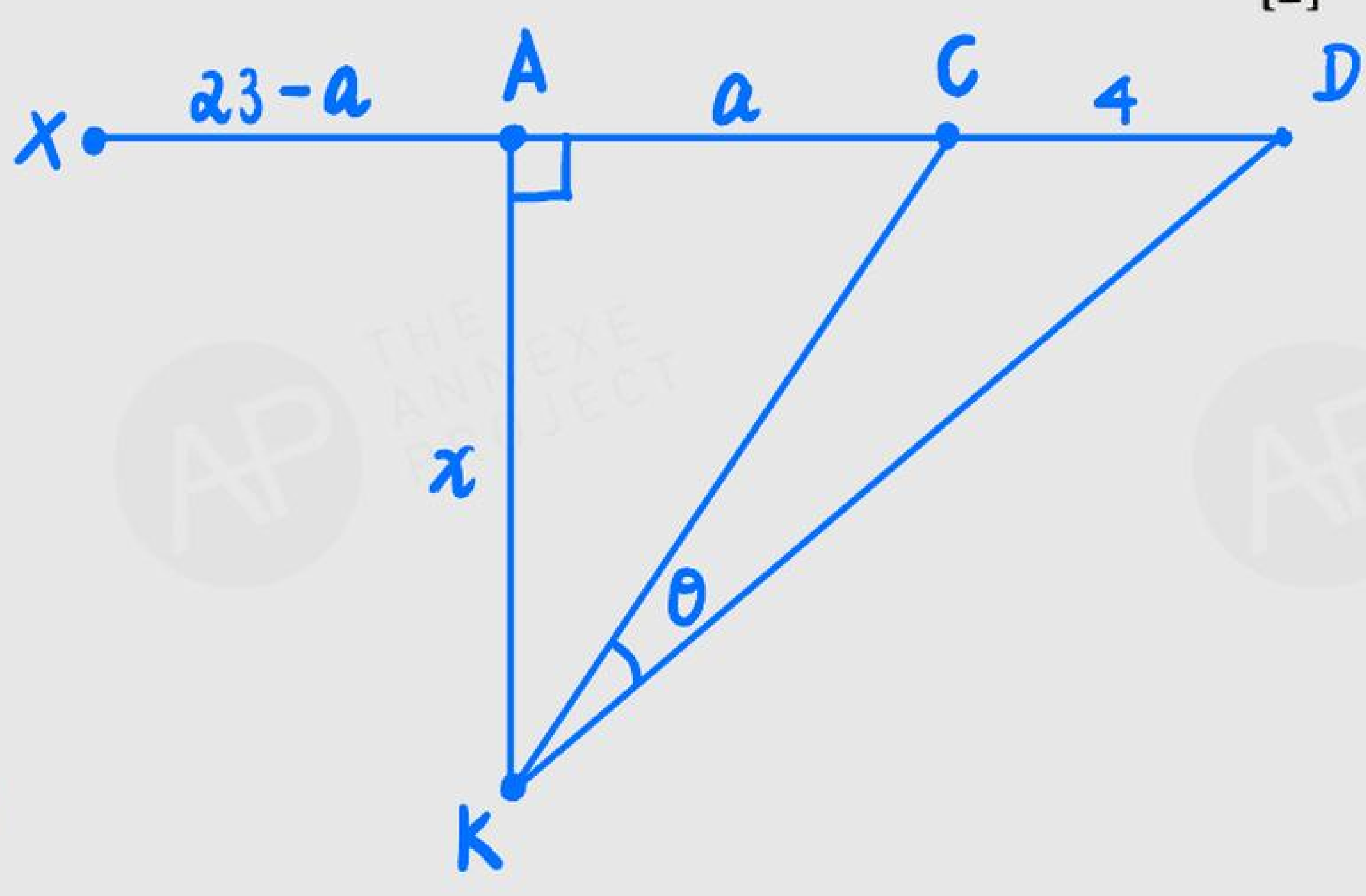
$\angle KDA = \frac{\pi}{4}$

(v) It is given that the length of the scoring line XY is 50 m. Find the range in which the optimal angle lies as the location of A varies between X and C. [2]

$2XC + 4 = 50$

$2XC = 46$

$XC = 23$ m



since $0 < a \leqslant 23$

$0.080086 \leqslant \theta < \frac{\pi}{2}$

$0.0801 \leqslant \theta < \frac{\pi}{2}$