1 The expression $7\cos\theta + 4\sin\theta$ is defined for $0 \leqslant \theta \leqslant \pi$ radians.

(i) Using $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, solve the equation $7\cos\theta + 4\sin\theta = 6$. [4]

$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha = 7\cos\theta + 4\sin\theta$

$R\cos\alpha = 7$ — (1)

$R\sin\alpha = 4$ — (2)

$\frac{(2)}{(1)}$: $\tan\alpha = \frac{4}{7}$

$\alpha = \tan^{-1}\frac{4}{7} = 0.51915$

$R = \sqrt{7^2 + 4^2} = \sqrt{65}$

$7\cos\theta + 4\sin\theta = 6$

$\sqrt{65}\cos(\theta - 0.51915) = 6$

$\cos(\theta - 0.51915) = \frac{6}{\sqrt{65}}$

$\theta - 0.51915 = 0.73145$ or 5.5517

$\therefore$ $\underline{\theta = 1.25}$ or 6.07 (Rej.)

(ii) State the largest and smallest values of $80 - (7\cos\theta + 4\sin\theta)^2$ and find the corresponding values of θ. [4]

$80 - (7\cos\theta + 4\sin\theta)^2 = 80 - 65[\cos(\theta - 0.51915)]^2$

Max. $80 - 65[\cos(\theta - 0.51915)]^2 = \underline{80}$

when $\theta - 0.51915 = \frac{\pi}{2}$

$\theta = 2.0899$

$= \underline{2.09}$ rad.

Min. $80 - 65[\cos(\theta - 0.51915)]^2 = \underline{15}$

when $\theta - 0.51915 = 0$

$\theta = \underline{0.519}$ rad.

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

2 (a) Solve the simultaneous equations

$$y = 2x^2 - 7, \quad (1)$$
$$y = 3x + 20. \quad (2)$$

[3]

Sub (2) into (1):

$3x + 20 = 2x^2 - 7$

$2x^2 - 3x - 27 = 0$

$(2x - 9)(x + 3) = 0$

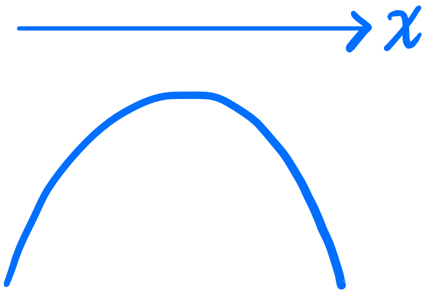
$\therefore x = \frac{9}{2}$ or -3

When $x = \frac{9}{2}$, $y = 3\left(\frac{9}{2}\right) + 20$
$= \frac{67}{2}$

When $x = -3$, $y = 11$

(b) Find the greatest value of the integer a for which $ax^2 + 5x - 2$ is negative for all x. [3]

firstly, $a < 0$.



secondly, the curve doesn't touch the x-axis,
i.e. no real roots.

Hence, $b^2 - 4ac < 0$

$25 - 4a(-2) < 0$

$8a < -25$

$a < -\frac{25}{8}$

$\therefore$ greatest value of integer $a = -4$

(c) Find the values of the constant c for which the line $y = 4x + c$ is a tangent to the curve $y = x^2 + cx + \frac{21}{4}$. [3]

Let $x^2 + cx + \frac{21}{4} = 4x + c$

$4x^2 + 4cx + 21 = 16x + 4c$

$4x^2 + (4c - 16)x + (21 - 4c) = 0$

$b^2 - 4ac = 0$

$(4c - 16)^2 - 4(4)(21 - 4c) = 0$

$16c^2 - 128c + 256 - 336 + 64c = 0$

$16c^2 - 64c - 80 = 0$

$c^2 - 4c - 5 = 0$

$(c - 5)(c + 1) = 0$

$\therefore \underline{c = -1 \text{ or } 5}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

3 (i) By considering the general term in the binomial expansion of $\left(\frac{3}{x^2}+x\right)^8$, explain why every term is dependent on x. [3]

$$T_{r+1} = \binom{8}{r}\left(\frac{3}{x^2}\right)^{8-r} x^r$$
$$= \binom{8}{r} 3^{8-r}(x^{-2})^{8-r} x^r$$
$$= \binom{8}{r} 3^{8-r} x^{-16+2r+r}$$
$$= \binom{8}{r} 3^{8-r} x^{-16+3r}$$

for a term to be independent,

$-16+3r=0$

$3r = 16$

$r = \frac{16}{3}$ (not a positive integer).

Hence, every term in the expansion is dependent on x.

(ii) Find the term independent of x in the expansion of $\left(\frac{3}{x^2}+x\right)^8(5-2x)$. [3]

Let $-16+3r = -1$

$3r = 15$

$r = 5$

$\therefore T_6 = \binom{8}{5} 3^{8-5} x^{-1}$

$= 56 \cdot 27 x^{-1}$

$= 1512 x^{-1}$

Hence, independent term in this expansion $= 1512 \times (-2)$

$= \underline{-3024}$

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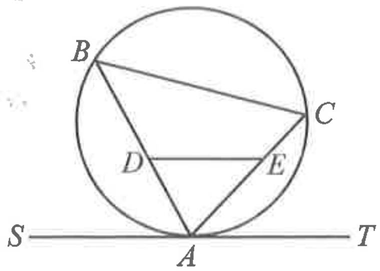
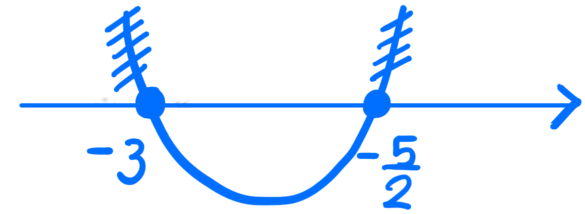
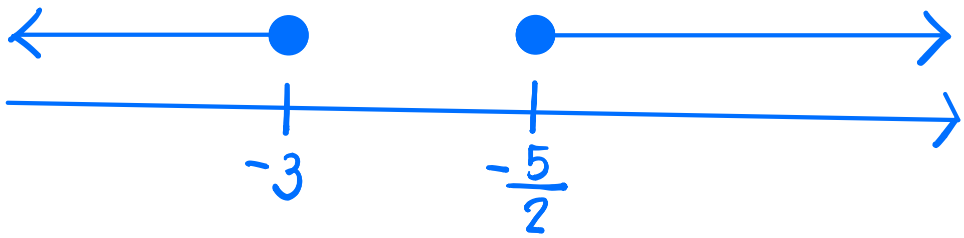


The diagram shows a circle passing through the points A, B and C. The straight line SAT is a tangent to the circle. The points D and E lie on AB and AC respectively and are such that $BCED$ is a cyclic quadrilateral. Prove that DE is parallel to ST. [5]

5 **(a)** Represent the solution set of $15(1+2x) \geqslant x(19-2x)$ on a number line. [4]

$$15(1+2x) \geqslant x(19-2x)$$
$$15+30x \geqslant 19x-2x^2$$
$$2x^2+30x-19x+15 \geqslant 0$$
$$2x^2+11x+15 \geqslant 0$$
$$(2x+5)(x+3) \geqslant 0$$

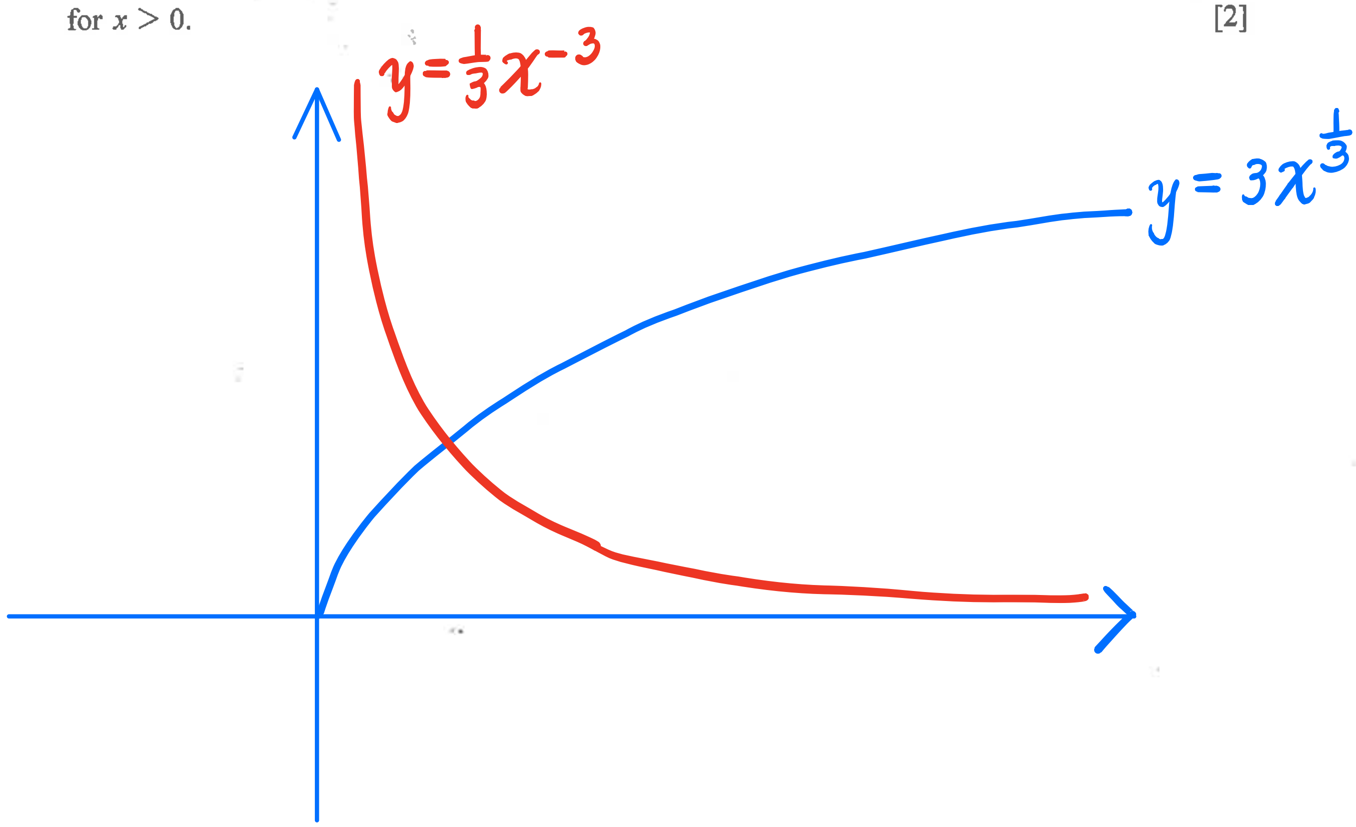




$$\underline{x \leq -3 \text{ or } x \geqslant -\frac{5}{2}}$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

(b) (i) Clearly labelling each graph, sketch, on the same axes, the graphs of $y = 3x^{\frac{1}{3}}$ and $y = \frac{1}{3}x^{-3}$ for $x > 0$. [2]



(ii) Show that the x-coordinate of the point of intersection of your graphs satisfies the equation $x^{10} = \frac{1}{729}$. [2]

Let $3x^{\frac{1}{3}} = \frac{1}{3x^3}$

$9x^{\frac{10}{3}} = 1$

$x^{\frac{10}{3}} = \frac{1}{9}$

$x^{10} = \left(\frac{1}{9}\right)^3$

$= \frac{1}{729}$

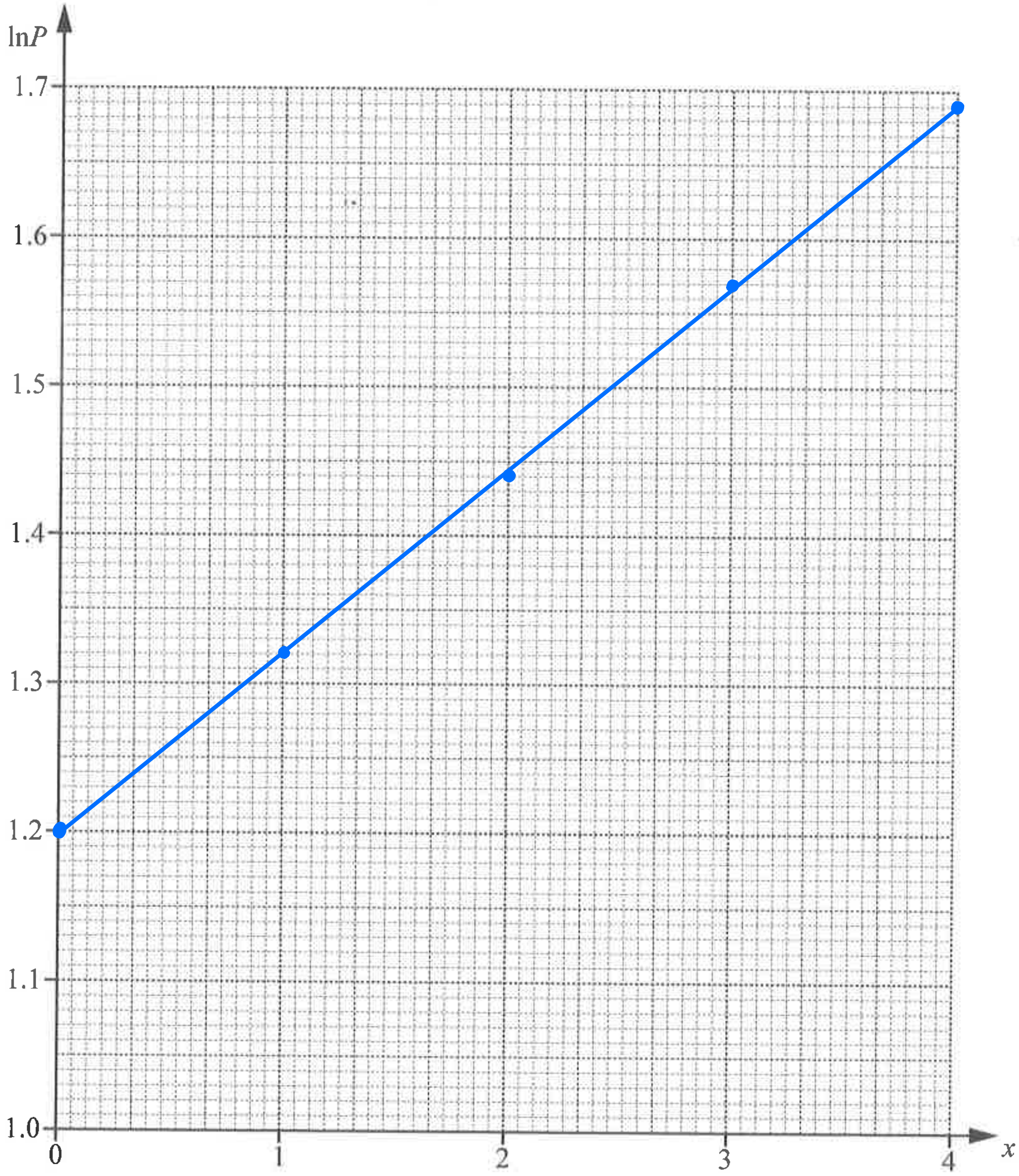
Hence, the x-coordinate of the point of intersection of both graphs satisfies $x^{10} = \frac{1}{729}$.

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

6 The table shows, to 3 significant figures, the population, P, in millions, of a country on January 1st at intervals of five years from 1995 to 2015. The variable x is measured in units of 5 years.

Year	1995	2000	2005	2010	2015
x	0	1	2	3	4
P	3.32	3.75	4.24	4.80	5.43
$\ln P$	1.20	1.32	1.44	1.57	1.69

(i) On the grid below plot $\ln P$ against x and draw a straight line graph. [2]



The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

(ii) Find the gradient of your straight line and hence express P in the form Ae^{kx}, where A and k are constants. [4]

$$\text{gradient} = \frac{1.69 - 1.20}{4 - 0}$$
$$= 0.1225 = \underline{0.123}$$

$$\ln P = 0.1225x + 1.2$$
$$P = e^{0.1225x + 1.2}$$
$$= e^{1.2}e^{0.1225x}$$
$$= \underline{3.32e^{0.123x}}$$

(iii) If this model for the population remains valid, find the first year of the interval in which the population first exceeds 8 million. [3]

Let $3.32e^{0.123x} > 8$

$$e^{0.123x} > \frac{8}{3.32}$$
$$0.123x > \ln\frac{8}{3.32}$$
$$x > 7.18$$
$$\therefore x = 8$$

i.e. $8 \times 5 = 40$ years.

$\Longrightarrow$ $\underline{2035}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

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7 (i) Show that $\frac{d}{dx}\left\{x(3x-5)^{\frac{5}{3}}\right\} = (8x-5)(3x-5)^{\frac{2}{3}}$. [5]

$$\frac{d}{dx}\,x(3x-5)^{\frac{5}{3}} = x \cdot \frac{5}{\cancel{3}}(3x-5)^{\frac{2}{3}}(\cancel{3}) + (3x-5)^{\frac{5}{3}}$$

$$= (3x-5)^{\frac{2}{3}}\left[5x + (3x-5)\right]$$

$$= \underline{(8x-5)(3x-5)^{\frac{2}{3}}}$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

(ii) Hence find $\int x(3x-5)^{\frac{2}{3}}\,dx$, giving your answer in the form $k(x+1)(3x-5)^{\frac{5}{3}}+c$, where k is a constant to be found and c is a constant of integration which cannot be found. [4]

$$\int x(3x-5)^{\frac{2}{3}}\,dx$$

$$= \frac{1}{8}\int 8x(3x-5)^{\frac{2}{3}}\,dx$$

$$= \frac{1}{8}\int (8x-5)(3x-5)^{\frac{2}{3}} + 5(3x-5)^{\frac{2}{3}}\,dx$$

$$= \frac{1}{8}\int (8x-5)(3x-5)^{\frac{2}{3}}\,dx + \frac{5}{8}\int (3x-5)^{\frac{2}{3}}\,dx$$

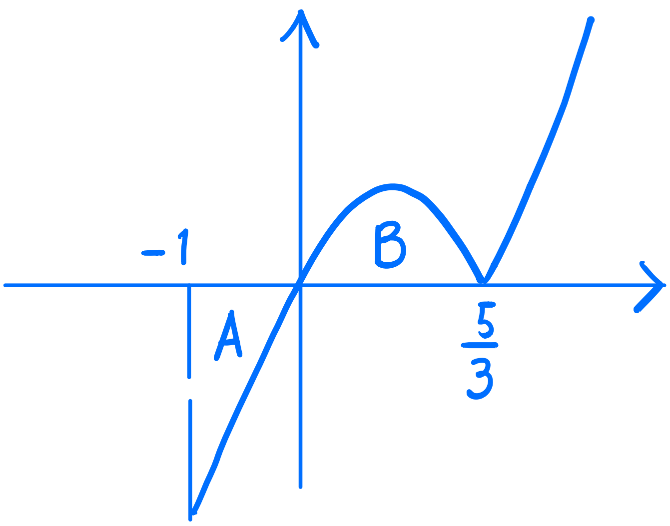
$$= \frac{1}{8}\left[x(3x-5)^{\frac{5}{3}}\right] + \frac{\cancel{5}}{8}\cdot\frac{(3x-5)^{\frac{5}{3}}}{\cancel{\left(\frac{5}{3}\right)}\cdot\cancel{3}} + C$$

$$= \underline{\frac{1}{8}(3x-5)^{\frac{5}{3}}(x+1) + C}$$

(iii) Find the value of $\int_{-1}^{\frac{5}{3}} x(3x-5)^{\frac{2}{3}}\,dx$ and explain what the result implies about the curve $y = x(3x-5)^{\frac{2}{3}}$. [2]

$$\int_{-1}^{\frac{5}{3}} x(3x-5)^{\frac{2}{3}}\,dx = \left[\frac{1}{8}(3x-5)^{\frac{5}{3}}(x+1)\right]_{-1}^{\frac{5}{3}}$$

$$= 0$$



This result implies that area A = area B.

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

8 (a) Solve the equation $e^x(1+e^x) = \frac{3}{4}$. [3]

Let $u = e^x$:

$u(1+u) = \frac{3}{4}$

$4u(1+u) = 3$

$4u^2 + 4u - 3 = 0$

$(2u-1)(2u+3) = 0$

$\therefore\ u = -\frac{3}{2}$ or $u = \frac{1}{2}$

i.e. $e^x = -\frac{3}{2}$ (No Solutions)

$e^x = \frac{1}{2}$

$x = \ln\frac{1}{2}$

$= \ln 1 - \ln 2$

$= -\ln 2$

(b) Solve the equation $1 + \log_2 y + \frac{1}{\log_8 2} = \log_2(y+3)$. [4]

$1 + \log_2 y + \log_2 8 = \log_2(y+3)$

$\log_2(y+3) - \log_2 y = 1 + \log_2 8$

$\log_2\left(\frac{y+3}{y}\right) = 1 + 3$

$\frac{y+3}{y} = 2^4$

$y + 3 = 16y$

$15y = 3$

$y = \frac{1}{5}$

(c) In order to obtain a graphical solution of the equation $x = \ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}$ a suitable straight line can be drawn on the same set of axes as the graph of $y = 3e^{\frac{x}{2}} + 4$. Make $e^{\frac{x}{2}}$ the subject of $x = \ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}$ and hence find the equation of this line. [3]

$$x = \ln\left(\frac{2x+7}{3}\right)^2$$

$$x = 2\ln\left(\frac{2x+7}{3}\right)$$

$$\frac{x}{2} = \ln\left(\frac{2x+7}{3}\right)$$

$$e^{\frac{x}{2}} = \frac{2x+7}{3}$$

$$\therefore\ 3e^{\frac{x}{2}} = 2x+7$$

$$3e^{\frac{x}{2}} + 4 = 2x + 11$$

Because the intersection of $y = 3e^{\frac{x}{2}} + 4$ and $y = 2x + 11$ gives the graphical solution of $x = \ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}$, the equation of the line to be added on the same axes would be $\underline{y = 2x + 11}$

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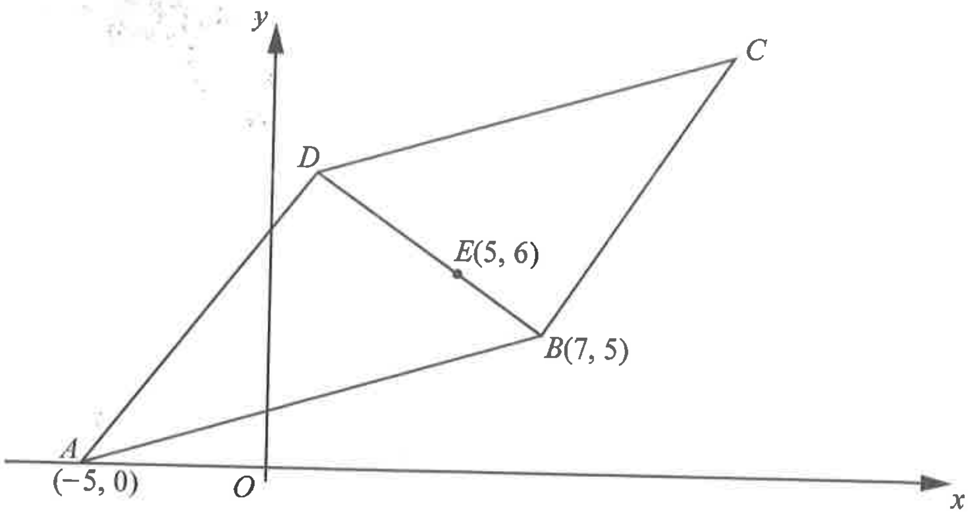
9



The diagram shows a parallelogram with vertices $A(-5, 0)$, $B(7, 5)$, C and D. The point $E(5, 6)$ lies on the diagonal BD of the parallelogram. The perimeter of the parallelogram is 46 units.

(i) Show that the x-coordinate of D satisfies the equation $5x^2 + 6x = 11$. [7]

gradient of $EB = \frac{6-5}{5-7} = -\frac{1}{2}$

$\therefore$ gradient of $DE = -\frac{1}{2}$

Let D be (x, y), then $\frac{y-6}{x-5} = -\frac{1}{2}$

$2y - 12 = 5 - x$

$2y = 17 - x$

$y = \frac{17-x}{2}$ —— ①

Length of $AB = \sqrt{(7+5)^2 + (5-0)^2}$

$= 13$ units.

Length of $AD = \sqrt{(x+5)^2 + (y-0)^2}$

$= \sqrt{(x+5)^2 + y^2}$

$2(13) + 2\sqrt{(x+5)^2 + y^2} = 46$

$\sqrt{(x+5)^2 + y^2} = 10$

$(x+5)^2 + y^2 = 100$ —— ②

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

Sub ① into ②: $(x+5)^2 + \left(\frac{17-x}{2}\right)^2 = 100$

$x^2+10x+25+\frac{289-34x+x^2}{4} = 100$

$4x^2+40x+100+289-34x+x^2 = 400$

$\underline{5x^2+6x=11}$ (Shown).

(ii) Determine the coordinates of D, explaining why the diagram is necessary. [3]

$5x^2+6x-11=0$

$(5x+11)(x-1)=0$

$x = \frac{-11}{5}$ or 1
(Rej.)

When $x=1$, $y=\frac{17-1}{2}$
$=8$

$\therefore \underline{D=(1,8)}$

The diagram is necessary because it shows that the x-coordinate of D is positive.

(iii) Find the coordinates of C. [1]

Coordinates of $C = (1+12, 8+5)$
$= \underline{(13, 13)}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

10 The function f is defined for $x \in \mathbb{R}$ and is such that $f''(x) = 48x^2 + 2e^{2x-1}$.

The line $x = \frac{1}{2}$ is the normal to the curve $y = f(x)$ at the point where $y = \frac{1}{4}$.

(i) Find an expression for $f'(x)$. [5]

$f''(x) = 48x^2 + 2e^{2x-1}$

$f'(x) = \int 48x^2 + 2e^{2x-1}\,dx$

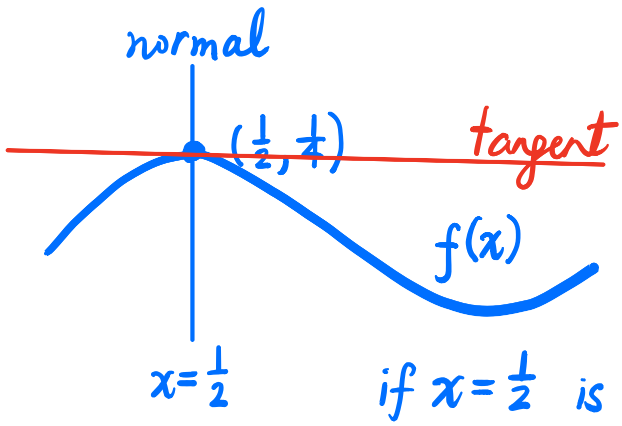
$= \frac{48x^3}{3} + \frac{2e^{2x-1}}{2} + C$

$= 16x^3 + e^{2x-1} + C$

$\therefore\ 0 = 16(\frac{1}{2})^3 + e^0 + C$

$C = -3$

hence, $\underline{f'(x) = 16x^3 + e^{2x-1} - 3}$



if $x = \frac{1}{2}$ is the normal to $y = f(x)$ where $y = \frac{1}{4}$, then gradient of tangent $= 0$ at $x = \frac{1}{2}$.

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

(ii) Hence find an expression for f(x). [4]

Since $f'(x) = 16x^3 + e^{2x-1} - 3$

then $f(x) = \int 16x^3 + e^{2x-1} - 3 \; dx$

$= \frac{16x^4}{4} + \frac{e^{2x-1}}{2} - 3x + c$

$= 4x^4 + \frac{1}{2}e^{2x-1} - 3x + c$

since $(\frac{1}{2}, \frac{1}{4})$ lies on $f(x)$,

$\frac{1}{4} = 4(\frac{1}{2})^4 + \frac{1}{2}e^0 - 3(\frac{1}{2}) + c$

$\therefore c = 1$

Hence, $\underline{f(x) = 4x^4 + \frac{1}{2}e^{2x-1} - 3x + 1}$

(iii) Show that the equation of the tangent to the curve $y = \mathrm{f}(x)$ at the point where the curve intersects the y-axis can be written as

$$2e(y + 3x - 1) = 2x + 1.$$

[3]

When the curve intersects the y-axis, $x = 0$

$f(x) = \frac{1}{2}e^{-1} + 1$

$= 1 + \frac{1}{2e}$

also, $f'(x) = e^{-1} - 3 = -3 + \frac{1}{e}$

$\therefore$ equation of tangent:

$y - (1 + \frac{1}{2e}) = (-3 + \frac{1}{e})(x - 0)$

$y - 1 - \frac{1}{2e} = -3x + \frac{x}{e}$

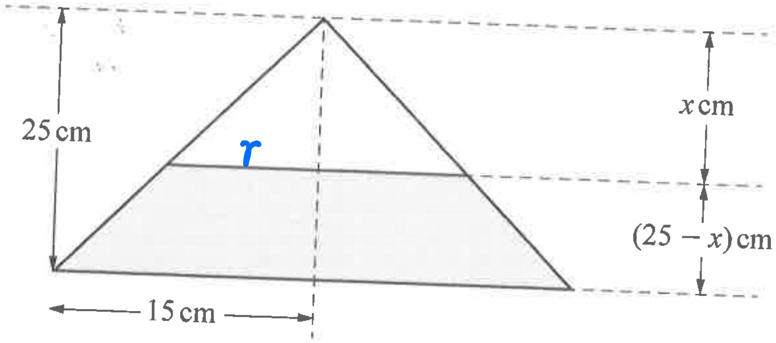
$2ey - 2e - 1 = -6ex + 2x$

$2ey + 6ex - 2e = 2x + 1$

$2e(y + 3x - 1) = 2x + 1$ (shown).

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

11 [The volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.]



The diagram shows a vertical cross-section of a container in the form of a cone, height 25 cm and base radius 15 cm. The container is initially empty. At time $t = 0$ liquid is allowed to flow in through a small hole close to the vertex. After t seconds the height of liquid in the container is $(25 - x)$ cm and the volume of liquid is $V \text{ cm}^3$.

(i) Show that $V = 3\pi\left(625 - \frac{x^3}{25}\right)$. [3]

Using similar Δs:

$$\frac{x}{r} = \frac{25}{15}$$

$$25r = 15x$$

$$r = \frac{3}{5}x$$

$$V = \frac{1}{3}\pi(15)^2(25) - \frac{1}{3}\pi r^2 x$$

$$= 1875\pi - \frac{1}{3}\pi\left(\frac{3}{5}x\right)^2 x$$

$$= 1875\pi - \frac{3}{25}\pi x^3$$

$$= 3\pi\left(625 - \frac{x^3}{25}\right) \quad \text{(shown).}$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

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Given that the rate of flow of the liquid is kx^2 cm^3/s, where k is a constant,

(ii) find an expression for $\frac{\mathrm{d}t}{\mathrm{d}x}$ in terms of π and k. [4]

Step 1: $\frac{dV}{dx} = -\frac{9}{25}\pi x^2$

2: Given $\frac{dV}{dt} = kx^2$

3: $\frac{dt}{dx} = \frac{dV}{dx} \cdot \frac{dt}{dV}$

$= \frac{-9\pi \cancel{x^2}}{25} \cdot \frac{1}{k\cancel{x^2}} = \frac{-9\pi}{25k}$

The liquid was allowed to flow for 72π seconds when the height of the liquid reached 12 cm.

(iii) By expressing t as a function of x, find the value of k. [4]

Since $\frac{dt}{dx} = \frac{-9\pi}{25k}$

$\therefore t = \frac{-9\pi}{25k}x + C$

at $t = 0$ s, $x = 25$ cm:

$\therefore 0 = \frac{-9\pi}{\cancel{25}k} \cdot \cancel{25} + C$

$C = \frac{9\pi}{k}$

Hence, $t = \frac{-9\pi}{25k}x + \frac{9\pi}{k}$

at $t = 72\pi$ s, $x = 13$ cm:

$\therefore 72\cancel{\pi} = \frac{-9\cancel{\pi}}{25k} \cdot 13 + \frac{9\cancel{\pi}}{k}$

$72 = \frac{108}{25k}$

$\therefore k = \frac{3}{50}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.