MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CENTRE NUMBER S

INDEX NUMBER

MATHEMATICS 9758/02

Paper 2

October/November 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

[Turn over

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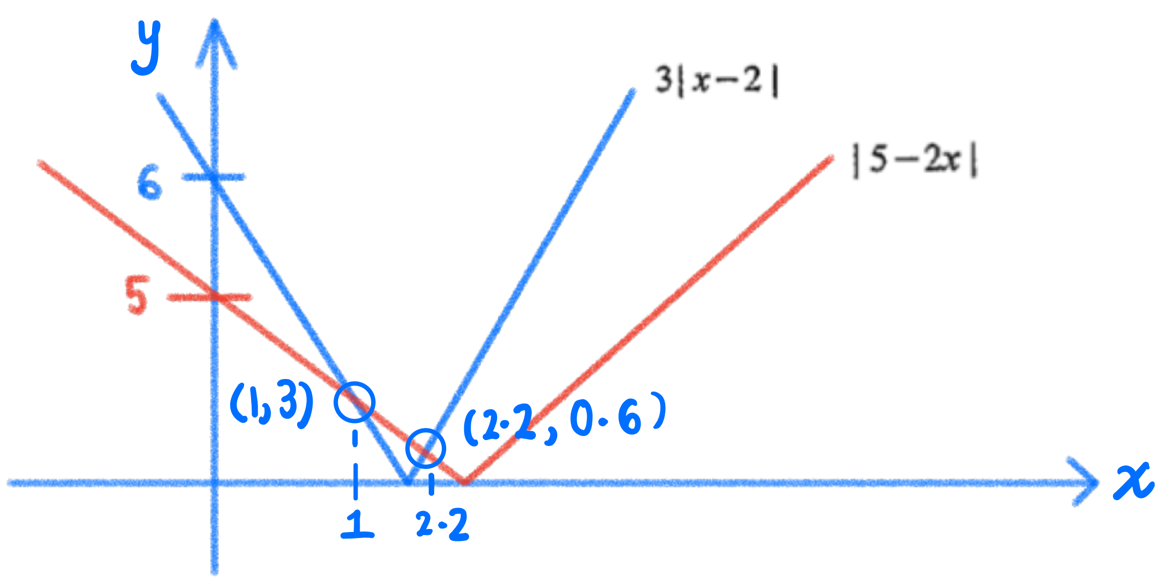
Section A: Pure Mathematics [40 marks]

1 (a) Find the set of values of x for which $3|x-2|<|5-2x|$. [2]

(b) Express $\dfrac{x+25}{x^2-4x-5}+3$ as a single simplified fraction. Hence, without using a calculator, solve exactly the inequality $\dfrac{x+25}{x^2-4x-5}>-3$. [4]

(a)



By GC: $\underline{1 < x < 2.2}$

(b)

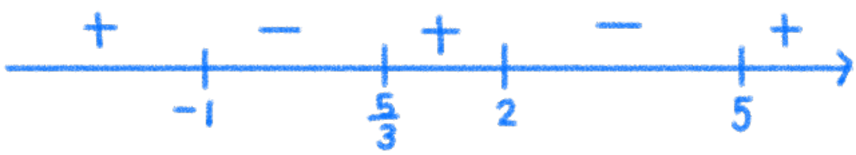
$$\frac{x+25}{x^2-4x-5}+3 = \frac{x+25+3(x^2-4x-5)}{x^2-4x-5}$$

$$= \frac{3x^2-11x+10}{x^2-4x-5}$$

Hence, $\dfrac{x+25}{x^2-4x-5}>-3$

$$\frac{3x^2-11x+10}{x^2-4x-5} > 0$$

$$\frac{(3x-5)(x-2)}{(x-5)(x+1)} > 0 \qquad x \neq -1,\ x \neq 5$$



$\therefore\ \underline{x<-1,\ \frac{5}{3}<x<2 \text{ or } x>5}$

2 (a) Given that $y = \ln(\sec x)$, show that $\frac{d^3y}{dx^3} = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)$. [3]

(b) Hence, or otherwise, obtain the Maclaurin expansion of y in terms of x up to and including the term in x^4. [3]

(c) By putting $x = \frac{1}{4}\pi$, find an approximation for $\ln 2$ in terms of π. [2]

(d) Using your answer to part (b), find an approximation to $\int_0^{\frac{1}{10}\pi} \ln(\sec x)\,dx$. Give your answer correct to 4 significant figures. [1]

(a) $y = \ln(\sec x)$

$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$

$\frac{d^2y}{dx^2} = \sec^2 x$

$\frac{d^3y}{dx^3} = 2(\sec x)^1 \cdot (\sec x \tan x)$

$= 2(\sec^2 x)(\tan x)$

$= 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$ (shown)

(b) $\frac{d^4y}{dx^4} = 2\frac{d^2y}{dx^2}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(2\frac{d^3y}{dx^3}\right)$

When $x = 0$:

$y = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 1$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 2$

$\therefore \ln(\sec x) = \frac{x^2}{2!}(1) + \frac{x^4}{4!}(2) + \cdots$

$= \frac{1}{2}x^2 + \frac{1}{12}x^4 + \cdots$

(c) When $x = \frac{\pi}{4}$

$\ln\left(\sec\frac{\pi}{4}\right) \approx \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \frac{1}{12}\left(\frac{\pi}{4}\right)^4$

$\therefore \ln\frac{1}{\left(\cos\frac{\pi}{4}\right)} = \frac{\pi^2}{32} + \frac{\pi^4}{3072}$

$\ln\sqrt{2} = \frac{\pi^2}{32} + \frac{\pi^4}{3072}$

$\frac{1}{2}\ln 2 = \frac{\pi^2}{32} + \frac{\pi^4}{3072}$

$\ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{1536}$

(d) $\int_0^{\frac{1}{10}\pi} \ln(\sec x)\,dx \approx \int_0^{\frac{\pi}{10}} \frac{1}{2}x^2 + \frac{1}{12}x^4$

$= 0.005218716 1$ (by GC)

$= \underline{0.005219}$ (4 s.f.)

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3 The points A and B have position vectors $\begin{pmatrix}-1\\2\\5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\8\end{pmatrix}$ respectively.

The point C is such that $\overrightarrow{BC} = 2\overrightarrow{AB}$.

(a) Find the position vector of C. [2]

The point D has position vector $\begin{pmatrix}1\\2\\d\end{pmatrix}$ and is such that $|\overrightarrow{AD}| = |\overrightarrow{BD}|$.

(b) Find the value of d. [2]

(c) Use a scalar product to find angle ADB. [3]

(d) Find exactly the position vector of the point P, where P is the centre of the circle that passes through A, B and D. [5]

(a)

$\overrightarrow{BC} = 2\overrightarrow{AB}$

$\overrightarrow{OC} - \overrightarrow{OB} = 2\overrightarrow{AB}$

$\overrightarrow{OC} = 2\overrightarrow{AB} + \overrightarrow{OB} = 2\left[\begin{pmatrix}1\\-2\\8\end{pmatrix} - \begin{pmatrix}-1\\2\\5\end{pmatrix}\right] + \begin{pmatrix}1\\-2\\8\end{pmatrix}$

$= \underline{\begin{pmatrix}5\\-10\\14\end{pmatrix}}$

(b)

$\overrightarrow{AD} = \begin{pmatrix}1\\2\\d\end{pmatrix} - \begin{pmatrix}-1\\2\\5\end{pmatrix} = \begin{pmatrix}2\\0\\d-5\end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix}1\\2\\d\end{pmatrix} - \begin{pmatrix}1\\-2\\8\end{pmatrix} = \begin{pmatrix}0\\4\\d-8\end{pmatrix}$

Given $|\overrightarrow{AD}| = |\overrightarrow{BD}|$

$\sqrt{4+(d-5)^2} = \sqrt{16+(d-8)^2}$

$\cancel{d^2} - 10d + 29 = \cancel{d^2} - 16d + 80$

$16d - 10d = 80 - 29$

$6d = 51$

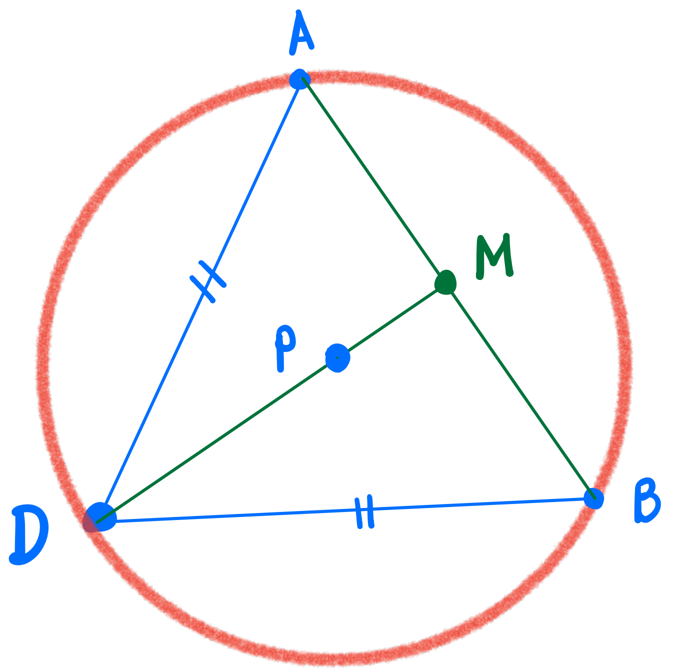
$\underline{d = 8.5}$

(c) Let angle $ADB = \theta$

$\theta = \cos^{-1}\dfrac{\overrightarrow{DA}\cdot\overrightarrow{DB}}{|\overrightarrow{DA}||\overrightarrow{DB}|} = \cos^{-1}\dfrac{\begin{pmatrix}-2\\0\\-3.5\end{pmatrix}\cdot\begin{pmatrix}0\\-4\\-0.5\end{pmatrix}}{\sqrt{2^2+3.5^2}\sqrt{4^2+0.5^2}}$

$$= \cos^{-1} \frac{1.75}{\sqrt{16.25}\sqrt{16.25}}$$

$$= \underline{83.8^\circ}$$



ⓓ. $\vec{OA} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$,

$\vec{OD} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix}$

Since $|\vec{AD}| = |\vec{BD}|$

$\triangle$ ADB is an isoceles $\triangle$.

The vertex D passing through P will land on M which is the midpoint of AB.

<u>Step 1</u>: Find $\vec{OM}$

$$\vec{OM} = \frac{\vec{OA} + \vec{OB}}{2} = \frac{\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}}{2} = \begin{pmatrix} 0 \\ 0 \\ 6.5 \end{pmatrix}$$

<u>Step 2</u>: Create the line DM

$$\vec{DM} = \begin{pmatrix} 0 \\ 0 \\ 6.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$l_{DM} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 6.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

<u>Step 3</u>: Express $\vec{OP}$

P lies on l_{DM}, hence $\vec{OP} = \begin{pmatrix} \lambda \\ 2\lambda \\ 6.5 + 2\lambda \end{pmatrix}$

<u>Step 4</u>: $|\vec{PA}| = |\vec{PD}|$

$$\vec{PA} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} \lambda \\ 2\lambda \\ 6.5 + 2\lambda \end{pmatrix} = \begin{pmatrix} -1-\lambda \\ 2-2\lambda \\ -1.5-2\lambda \end{pmatrix}$$

$$\vec{PD} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} - \begin{pmatrix} \lambda \\ 2\lambda \\ 6.5 + 2\lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda \\ 2-2\lambda \\ 2-2\lambda \end{pmatrix}$$

$$\left|\begin{pmatrix} -1-\lambda \\ 2-2\lambda \\ -1.5-2\lambda \end{pmatrix}\right| = \left|\begin{pmatrix} 1-\lambda \\ 2-2\lambda \\ 2-2\lambda \end{pmatrix}\right|$$

$$\sqrt{(-1-\lambda)^2+(2-2\lambda)^2+(-1.5-2\lambda)^2} = \sqrt{(1-\lambda)^2+(2-2\lambda)^2+(2-2\lambda)^2}$$

$$1+2\lambda+\cancel{\lambda^2}+4-8\lambda+\cancel{4\lambda^2}+2.25+6\lambda+\cancel{4\lambda^2} = 1-2\lambda+\cancel{\lambda^2}+4-8\lambda+\cancel{4\lambda^2}+4-8\lambda+\cancel{4\lambda^2}$$

$$7.25 = 9-18\lambda$$

$$18\lambda = 1.75$$

$$\lambda = \frac{7}{72}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} \frac{7}{72} \\ 2\left(\frac{7}{72}\right) \\ 6.5+2\left(\frac{7}{72}\right) \end{pmatrix} = \underline{\begin{pmatrix} \frac{7}{72} \\ \frac{7}{36} \\ \frac{241}{36} \end{pmatrix}}$$

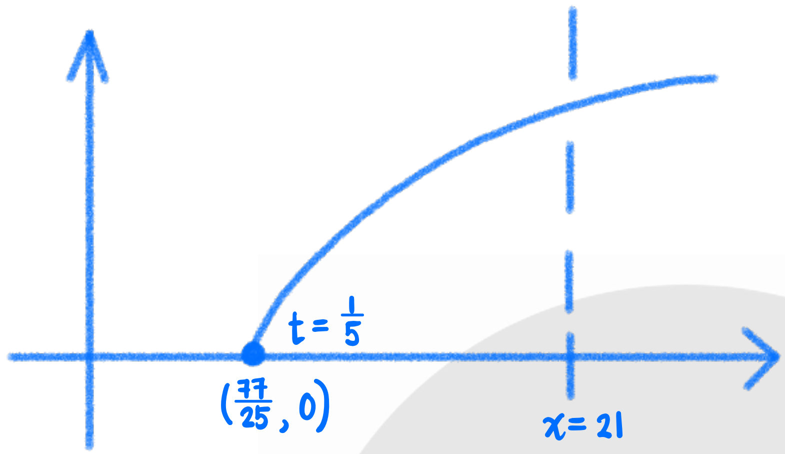
4 The curve C is defined by the parametric equations $x = 2t^2 + 3$ and $y = 5t - 1$, where $t \geqslant \frac{1}{5}$.

(a) Find the exact area between the curve C, the x-axis and the line $x = 21$. [3]

The curve D is defined by the parametric equations $x = 5u$ and $y = \frac{4}{u}$, where $u \neq 0$.

(b) Find a point of intersection, A, of the curves C and D. Show that there are no other points of intersection. [5]

(c) Find the coordinates of the point where the tangent to curve C at the point A meets curve D for a second time. [5]



ⓐ When $x = 21$, $2t^2 + 3 = 21$

$t^2 = 9$

$t = 3$ (since $t \geqslant \frac{1}{5}$)

$x = 2t^2 + 3$

$\frac{dx}{dt} = 4t$

$$\int_{t=\frac{1}{5}}^{t=3} (5t-1)(4t)\,dt$$

$$= \int_{\frac{1}{5}}^{3} 20t^2 - 4t \; dt$$

$$= \left[\frac{20t^3}{3} - 2t^2 \right]_{\frac{1}{5}}^{3}$$

$$= 162 - \left(-\frac{2}{75}\right)$$

$$= \frac{12152}{75} \text{ sq. units}$$

ⓑ Cartesian equation of D:

$x = 5u$ —— ① $\quad y = \frac{4}{u}$ —— ②

from ①: $u = \frac{x}{5}$ —— ③

Sub ③ into ②: $y = \frac{4}{\left(\frac{x}{5}\right)} = \frac{20}{x}$

Sub $y = \frac{20}{x}$ into eqn of C:

$\therefore \quad 5t-1 = \frac{20}{2t^2+3}$

$(5t-1)(2t^2+3) = 20$

$10t^3 - 2t^2 + 15t - 23 = 0$

By GC: $t = 1$

Since there is only 1 real solution of t, there is only 1 intersection point between C and D.

Sub $t=1$ into $y = 5t-1$ and $x = 2t^2+3$:

$x = 5$, $y = 4$. $\quad A = (5, 4)$

(c) $x = 2t^2+3 \quad ; \quad y = 5t-1$

$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 5$

$\therefore \frac{dy}{dx} = \frac{5}{4t}$

When $t = 1$, $\frac{dy}{dx} = \frac{5}{4}$

Eqn. of tangent to curve C at point A:

$y - 4 = \frac{5}{4}(x-5)$

$y = \frac{5}{4}x - \frac{9}{4}$ —— (4)

Eqn. of curve D: $y = \frac{20}{x}$ —— (5)

Sub (5) into (4): $\frac{20}{x} = \frac{5}{4}x - \frac{9}{4}$

$80 = 5x^2 - 9x$

$5x^2 - 9x - 80 = 0$

$(x-5)(5x+16) = 0$

$x = 5$ (rej.) or $x = \frac{-16}{5}$

When $x = \frac{-16}{5}$, $y = 20 \div \frac{-16}{5}$

$= 20 \times \frac{-5}{16} = \frac{-25}{4}$. The point is $\left(\frac{-16}{5}, \frac{-25}{4}\right)$

Section B: Probability and Statistics [60 marks]

5 A child's toy has 36 slots, numbered from 1 to 36. A child puts a ball into the toy, the ball falls into one of the 36 slots and the child's score is the number of that slot. The ball is equally likely to fall into any one of the slots.

Sadiq is investigating four possible events, A, B, C and D, which are defined as follows.

A	the score is odd
B	the score is even
C	the score is a multiple of 3
D	the score is a multiple of 6

(a) (i) State which pairs of the events, if any, are mutually exclusive. [1]

(ii) Show that A and C are independent events, and state another pair of independent events. [2]

Sadiq notices that a ball has become stuck in the slot labelled 36, and so balls put into the toy are now falling, with equal probability, into one of only 35 slots, and the score can only be from 1 to 35.

(b) (i) State which pairs of the four events, if any, are **now** mutually exclusive. [1]

(ii) Determine whether A and C are **now** independent events. [1]

a i. A and B are mutually exclusive.
A and D are mutually exclusive.

ii. $P(A) = \frac{18}{36} = \frac{1}{2}$

$P(C) = \frac{12}{36} = \frac{1}{3}$

$P(A \cap C) = \{3, 9, 15, 21, 27, 33\} = \frac{6}{36} = \frac{1}{6}$

$P(A) \cdot P(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Since $P(A \cap C) = P(A) \cdot P(C) = \frac{1}{6}$,
A and C are independent events.

$\Rightarrow$ B and C are also independent events.

b i. A and B, A and D are still mutuall exclusive.

ii. $P(A) = \frac{18}{35}$, $P(C) = \frac{11}{35}$, $P(A) \cdot P(C) = \frac{18}{35} \times \frac{11}{35} = \frac{198}{1225}$

$P(A \cap C) = \{3, 9, 15, 21, 27, 33\} = \frac{6}{35}$

Since $P(A \cap C) \neq P(A) \cdot P(C)$,
A and C are not independent events now.

6 A bag contains r red counters and b blue counters, where $r > 12$ and $b > 12$. Mei randomly removes 12 counters from the bag. The probability that there are 4 red counters among Mei's 12 counters is the same as the probability that there are 3 red counters.

(a) Show that $9r + 5 = 4b$. [2]

The probability that there are 3 red counters among Mei's 12 counters is $\frac{5}{3}$ times the probability that there are 2 red counters.

(b) Derive an equation similar to the equation in part (a) and hence find the probability that just one of the 12 counters removed is red. [6]

(a)
$$\frac{{}^rC_4 \times {}^bC_8}{{}^{r+b}C_{12}} = \frac{{}^rC_3 \times {}^bC_9}{{}^{r+b}C_{12}}$$

$${}^rC_4 \times {}^bC_8 = {}^rC_3 \times {}^bC_9$$

$$\frac{\cancel{r!}}{(r-4)!\,4!} \times \frac{\cancel{b!}}{(b-8)!\,8!} = \frac{\cancel{r!}}{(r-3)!\,3!} \times \frac{\cancel{b!}}{(b-9)!\,9!}$$

$$\frac{(r-3)!\,3!}{(r-4)!\,4!} = \frac{(b-8)!\,8!}{(b-9)!\,9!}$$

$$\frac{(r-3)\cancel{(r-4)!}\cancel{3!}}{\cancel{(r-4)!}\,4\times\cancel{3!}} = \frac{(b-8)\cancel{(b-9)!}\,\cancel{8!}}{\cancel{(b-9)!}\,9\times\cancel{8!}}$$

$$\frac{r-3}{4} = \frac{b-8}{9}$$

$$9r - 27 = 4b - 32$$

$$9r + 5 = 4b \quad \text{——} \quad (1)$$

Take Note:
$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$
is in the formula booklet.

Also:
$$(r-3)! = (r-3)(r-4)!$$

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(b) Given
$$\frac{{}^rC_3 \times {}^bC_9}{\cancel{{}^{r+b}C_{12}}} = \frac{5}{3} \times \frac{{}^rC_2 \times {}^bC_{10}}{\cancel{{}^{r+b}C_{12}}}$$

$$\frac{\cancel{r!}}{(r-3)!\,3!} \times \frac{\cancel{b!}}{(b-9)!\,9!} = \frac{5}{3} \times \frac{\cancel{r!}}{(r-2)!\,2!} \times \frac{\cancel{b!}}{(b-10)!\,10!}$$

$$\frac{(r-2)!\,\cancel{2!}\times\cancel{3}}{(r-3)!\,\cancel{3!}} = \frac{(b-9)!\,9!\times 5}{(b-10)!\,10!}$$

$$(r-2) = \frac{(b-9)\times\cancel{5}}{\cancel{10}\,2}$$

$$2r - 4 = b - 9$$

$$b = 2r + 5 \quad \text{——} \quad (2)$$

Sub 2 into 1:

$$9r + 5 = 4(2r+5)$$
$$9r - 8r = 20 - 5$$
$$\underline{r = 15} \quad \text{and} \quad b = 2(15) + 5$$
$$\underline{b = 35}$$

$$P(\text{1 out of 12 counters is red}) = \frac{{}^{15}C_1 \times {}^{35}C_{11}}{{}^{50}C_{12}}$$
$$= 0.051552$$
$$= \underline{0.0516}$$

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7 The numbers of mobile phone subscriptions in Singapore, y million, for certain years from 2004 are given in the following table. The variable x is the number of years after a base year of 2000.

Year	2004	2006	2008	2010	2012	2014	2016	2018
Number of years after base year, x	4	6	8	10	12	14	16	18
Number of subscriptions, y million	3.99	4.79	6.41	7.38	8.07	8.10	8.46	8.39

Ling thinks that the number of mobile phone subscriptions in Singapore can be modelled by one of the formulae

$$y = ax + b, \qquad e^y = cx + d,$$

where a, b, c and d are constants.

(a) Find, correct to 4 decimal places, the value of the product moment correlation coefficient

(i) between x and y, [1]

(ii) between x and e^y. [1]

(a) i. By GC : $r = 0.9281271487 = \underline{0.9281 \text{ (4 d.p.)}}$

ii. By GC : $r = 0.9697235231 = \underline{0.9697 \text{ (4 d.p.)}}$

(b) Explain which of Ling's models, $y = ax + b$ or $e^y = cx + d$, gives a better fit to the data and find the equation of the regression line for this model. [3]

(c) Use the equation of the regression line to estimate the number of mobile phone subscriptions in 2024. Explain whether your estimate is reliable. [2]

(b) Since $|r|$ between x and e^y is closer to 1, it suggests a stronger linear correlation between x and e^y.
Hence, $e^y = cx + d$ is a better fit.

By GC, $e^y = -1881.481827 + 375.6152263x$

$\underline{e^y = -1880 + 376x}$

(c) When $x = 24$, $e^y = -1881.481827 + 375.6152263(24)$

$\therefore y = 8.8725$

$= \underline{8.87 \text{ millions}}$

Since $x = 24$ is outside of the range of data collected (extrapolation), the estimate is <u>not reliable</u>.

8 (a) A company has a new machine designed to fill bags with, on average, 1 kg of granulated sugar. The production manager wishes to investigate if the machine is adjusted correctly. He intends to take a sample of bags and carry out a hypothesis test.

(i) State null and alternative hypotheses for the manager's test, defining any parameters you use. [2]

The production manager decides to take the first bag of sugar produced each morning and the first bag of sugar produced each afternoon, in a 5-day working week, to form a sample of 10 bags for the test.

(ii) Give **two** reasons why the production manager's sample is not suitable for a z-test. [2]

a (i) Let μ be the population mean mass of sugar in kg.

$H_0: \mu = 1$ kg against

$H_1: \mu \neq 1$ kg.

where H_0 = null hypothesis,

H_1 = alternative hypothesis.

(ii) 1. Since the distribution is unknown, we will have to apply Central Limit Theorem to approximate the distribution to be normal. This is because z-test can only be conducted if the distribution is normal. However, sample size should be large enough ($n \geqslant 30$) so that CLT can be applied. Since $n = 10$, z-test is not suitable.

2. Since one bag is chosen in the morning and one in the afternoon, the probability of each bag selected is not independent of one another. Hence, the sample is not random.

(b) The company has a different machine which fills larger bags with, on average, 2 kg of granulated sugar. One of the company's sales representatives has reported that some customers suspect the machine is no longer set correctly, and that the average mass of sugar in the bags may in fact be less than 2 kg. The production manager decides to carry out a hypothesis test at the 2.5% level of significance with a suitable sample of 40 bags of sugar. Summary data for the mass, x kg, of sugar in these bags is as follows.

$$n = 40 \qquad \Sigma x = 78.88 \qquad \Sigma x^2 = 155.6746$$

(i) State the hypotheses and find the critical region for this test. [5]

(ii) State the conclusion of the test in the context of the question. [2]

(i) Step 1 : $\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{78.88}{40} = 1.972$

$s^2 = \dfrac{1}{n-1}\left[\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right]$

$= \dfrac{1}{39}\left[155.6746 - \dfrac{78.88^2}{40}\right] = 0.00316$

Step 2 :

Let μ be the population mean mass of the sugar in the larger bag in kg.

To test $H_0 : \mu = 2$ against

$H_1 : \mu < 2$ at 2.5% level of significance.

Step 3 :

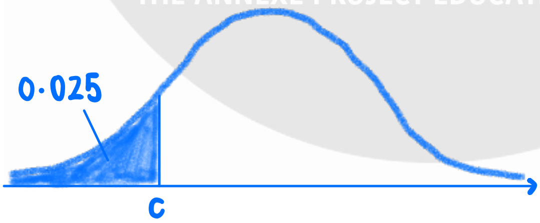
Since n is large, by CLT :

$\bar{X} \sim N\left(2, \dfrac{0.00316}{40}\right)$ approximately.

Test Statistic :

$Z = \dfrac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$

$= \dfrac{\bar{X} - 2}{\sqrt{\frac{0.00316}{40}}} = 112.50879(\bar{X} - 2)$



$P(Z \leq c) = 0.025$

$c = -1.95996$

In order for H_0 to be rejected, z-value $\leq c$

$112.50879(\bar{x} - 2) \leq -1.95996$

$\bar{x} - 2 \leq -0.017421$

$\bar{x} \leq 1.9826$

Since $\bar{x} > 0$,

$\therefore$ critical region is $0 < \bar{x} \leq 1.9826$

(ii) Since $\bar{x} = 1.972$ (from part i), H_0 is rejected.

We conclude that there is sufficient evidence at 2.5% level of significance to say that the machine is no longer set correctly and the average mass of sugar is in fact less than 2 kg.

9 In this question, you should state the parameters of any distributions you use.

A company produces wooden planks of two different lengths, Regular and Long. The lengths, in metres, of the Long planks follow the distribution $N(1.82, 0.2^2)$.

(a) Find the probability that the length of a randomly chosen Long plank is less than 1.79 m. [1]

(b) Find the probability that the total length of 8 randomly chosen Long planks is greater than 14.5 m. [3]

The lengths, in metres, of the Regular planks follow the distribution $N(1.22, 0.3^2)$.

(c) Sylvio buys 120 of the Regular planks. Calculate the expected number of these planks that are longer than 1.25 m. [2]

(d) Find the probability that the total length of 10 randomly chosen Long planks differs by less than 0.65 m from the total length of 16 randomly chosen Regular planks. [3]

(a) Let L be the length of a Long plank in metres.

$$L \sim N(1.82, 0.2^2)$$

$P(L < 1.79) = 0.44038$

$= \underline{0.440}$

(b) $L_1 + L_2 + L_3 + \cdots + L_8 \sim N(8 \times 1.82, 8 \times 0.2^2)$

$P(L_1 + L_2 + \cdots + L_8 > 14.5) = 0.54224$

$= \underline{0.542}$

(c) Let R be the length of a regular plank in metres.

$$R \sim N(1.22, 0.3^2)$$

$P(R > 1.25) = 0.46017$

Let X be no. of planks out of 120 that are longer than 1.25 m.

$$X \sim B(120, 0.46017)$$

$E(X) = np = 120 \times 0.46017 = 55.2$

$\approx$ __55 planks.__

(d) $E(L_1+L_2+\cdots+L_{10}-R_1-R_2-\cdots R_{16}) = (10\times1.82)-(16\times1.22)$
$= -1.32$

$Var(L_1+L_2+\cdots+L_{10}-R_1-R_2-\cdots R_{16}) = (10\times0.2^2)+(16\times0.3^2)$
$= 1.84$

$\therefore L_1+L_2+\cdots+L_{10}-R_1-R_2-\cdots R_{16} \sim N(-1.32, 1.84)$

$P(|L_1+L_2+\cdots+L_{10}-R_1-R_2-\cdots R_{16}| < 0.65)$

$= P(-0.65 < L_1+L_2+\cdots+L_{10}-R_1-R_2-\cdots R_{16} < 0.65)$

$= 0.23747$

$= \underline{0.237}$

The company finds that there is a demand for Short planks. These planks are produced by cutting the Long planks into three exactly equal lengths, or the Regular planks into two exactly equal lengths.

(e) Find the probability that the length of a randomly chosen Short plank made from a Long plank is greater than the length of one made from a Regular plank. You should ignore any wastage caused by cutting the plank. [4]

(f) Without doing any detailed calculation, explain how your answer to part (e) would change if each cut of a plank caused a small amount of wastage. [1]

(e) $E\left(\frac{L}{3}-\frac{R}{2}\right) = \frac{1}{3}\times1.82 - \frac{1}{2}\times1.22 = \frac{-1}{300}$

$Var\left(\frac{L}{3}-\frac{R}{2}\right) = \frac{1}{9}\times0.2^2 + \frac{1}{4}\times0.3^2 = \frac{97}{3600}$

$\therefore \frac{L}{3}-\frac{R}{2} \sim N\left(\frac{-1}{300}, \frac{97}{3600}\right)$

$P\left(\frac{L}{3} > \frac{R}{2}\right) = P\left(\frac{L}{3}-\frac{R}{2} > 0\right) = 0.49190$
$= \underline{0.492}$

(f) 2 cuts of the long plank should produce more wastage than 1 cut of the regular plank.
Hence $P\left(\frac{L}{3} > \frac{R}{2}\right)$ should be less than 0.492.

10 (a) A small company makes 50 glass ornaments each working day. Some of the ornaments turn out to be faulty.

(i) State, in the context of the question, two assumptions needed for the number of faulty ornaments made in a day to be well modelled by a binomial distribution. [2]

Assume now that the number of faulty ornaments produced each day has the distribution $B(50, 0.04)$.

(ii) Show that the numerical values of the mean and variance of this distribution differ by 0.08. [1]

(iii) Find the probability that no more than 2 faulty ornaments are produced on a randomly chosen working day. [1]

(iv) Find the probability that no more than 2 faulty ornaments are produced on at least 3 days in a randomly chosen 5-day working week. State the distribution you use. [3]

(v) Find the probability that no more than 10 faulty items are produced in a randomly chosen 5-day working week. State the distribution you use. [2]

(i) The event that a glass ornament is faulty or not occurs independently of other glass ornaments.

The probability of a glass ornament being faulty is constant for each glass ornament.

(ii) Let X be the r.v. denoting the number of faulty ornaments out of 50 glass ornaments each day.

$$X \sim B(50, 0.04)$$

$E(X) = 50 \times 0.04 = 2$, $Var(X) = 50 \times 0.04 \times (0.96) = 1.92$

$E(X) - Var(X) = 2 - 1.92 = 0.08$.

(iii) $P(X \leq 2) = 0.67671 = 0.677$

(iv) Let Y be the r.v. denoting no. of days out of 5, where there are no more than 2 faulty ornaments in a day.

$$Y \sim B(5, 0.67671)$$

$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.80477 = 0.805$

(v) Let W be the r.v. denoting no. of faulty ornaments out of 250.

$$W \sim B(250, 0.04)$$

$P(W \leq 10) = 0.58306 = 0.583$

(b) The company also makes pens which are sold in randomly packed boxes of one hundred pens. The probability of a pen being **not** faulty is p, where $0 < p < 1$.

For quality control purposes, a random sample of pens from each box is tested. Mr Lu and Mrs Ming carry out the tests but they use different methods.

Mr Lu tests a random sample of 6 pens from a box. If there are no faulty pens or only 1 faulty pen the box is accepted.

Mrs Ming tests a random sample of 3 pens from a box.

- If there are no faulty pens in her sample the box is accepted.
- If there are 2 or 3 faulty pens in her sample the box is rejected.
- If there is 1 faulty pen in her sample she takes a second random sample of 3 pens. She accepts the box if there are no faulty pens in this second sample.

Show algebraically that Mrs Ming accepts a greater proportion of boxes than Mr Lu does. [6]

Check on Mr Lu:

Let L be the r.v. denoting no. of non-faulty pens in a sample of 6.

$$L \sim B(6, p)$$

$$\begin{aligned} P(\text{Mr Lu accepts the box}) &= P(L \geq 5) \\ &= P(L=5) + P(L=6) \\ &= {}^6C_5\, p^5(1-p) + {}^6C_6\, p^6 \\ &= 6p^5(1-p) + p^6 \\ &= p^5(6-6p+p) \\ &= \underline{p^5(6-5p)} \end{aligned}$$

Check on Mrs Ming:

Let M be the r.v. denoting no. of non-faulty pens in a sample of 3.

$$M \sim B(3, p)$$

$$\begin{aligned} P(\text{Mrs Ming accepts the box}) &= P(M=3) + P(M=2) \cdot P(M=3) \\ &= {}^3C_3\, p^3 + {}^3C_2\, p^2(1-p) \cdot {}^3C_3\, p^3 \\ &= p^3 + 3p^2(1-p)p^3 \\ &= p^3(1+3p^2-3p^3) \end{aligned}$$

$$\begin{aligned} p^3(1+3p^2-3p^3) - p^5(6-5p) &= p^3 + 3p^5 - 3p^6 - 6p^5 + 5p^6 \\ &= 2p^6 - 3p^5 + p^3 \\ &= p^3(2p^3 - 3p^2 + 1) \end{aligned}$$

$= p^3(2p+1)(p-1)^2$ $\boxed{\text{using GC}}$

> 0

Explanation:

Since $0<p<1$, $p^3>0$, $(p-1)^2>0$ and $(2p+1)>0$

Hence, P(Mrs Ming accepts the box) − P(Mr Lu accepts the box) > 0

$\therefore$ P(Mrs Ming accepts the box) > P(Mr Lu accepts the box)

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