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MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CENTRE NUMBER S

INDEX NUMBER

* 4 3 1 5 4 0 3 9 1 4 *

MATHEMATICS 9758/01

Paper 1

October/November 2024

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **22** printed pages and **2** blank pages.

Singapore Examinations and Assessment Board

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1 The graph of $y = \dfrac{x+1}{ax^2+bx+c}$, where a, b and c are non-zero constants, has an asymptote at $x = -\frac{1}{2}$.
The graph also has a turning point at $\left(-2, -\frac{1}{9}\right)$.

Find the values of a, b and c. [4]

Sub $(-2, -\frac{1}{9})$ into equation of curve:

$$-\frac{1}{9} = \frac{-2+1}{a(-2)^2+b(-2)+c}$$

$$\frac{-1}{9} = \frac{-1}{4a-2b+c}$$

$$4a - 2b + c = 9 \quad \text{——— } (1)$$

Vertical asymptote $x = -\frac{1}{2}$ indicates that

$$a\left(-\tfrac{1}{2}\right)^2 + b\left(-\tfrac{1}{2}\right) + c = 0$$

$$\tfrac{1}{4}a - \tfrac{1}{2}b + c = 0 \quad \text{——— } (2)$$

$$\frac{dy}{dx} = \frac{(ax^2+bx+c)(1) - (x+1)(2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{ax^2 + \cancel{bx} + c - 2ax^2 - \cancel{bx} - 2ax - b}{(ax^2+bx+c)^2}$$

$$= \frac{-ax^2 - 2ax + c - b}{(ax^2+bx+c)^2}$$

When $x = -2$, $\frac{dy}{dx} = 0$:

$$-a(-2)^2 - 2a(-2) + c - b = 0$$

$$-4a + 4a - b + c = 0$$

$$-b + c = 0 \quad \text{——— } (3)$$

By GC:

$a = 2$, $b = -1$, $c = -1$

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2 Two vectors **a** and **b** are such that $|\mathbf{a}\times\mathbf{b}| = 3$. It is given that **a** is a unit vector and $\mathbf{b}.\mathbf{b} = 9$.

Show that **a** and **b** are perpendicular. [3]

Given $\underset{\sim}{b}\cdot\underset{\sim}{b} = 9$

$\therefore \quad |\underset{\sim}{b}|^2 = 9 \implies |\underset{\sim}{b}| = 3$

Given $|\underset{\sim}{a}\times\underset{\sim}{b}| = 3$

$\therefore \quad |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta = 3$

$1\times 3\sin\theta = 3$

$\sin\theta = 1$

$\underline{\theta = 90^\circ}$

Hence, $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$.

3 The equation $z^3 + az^2 + bz + c = 0$, where a, b and c are constants, has roots 3 and w where w is a complex number.

(a) State a condition on a, b and c for the third root to be w^*. [1]

a, b and $c \in \mathbb{R}$

(b) Given that the condition in part **(a)** holds, and that $w = -1 + 2\mathrm{i}$, find the values of a, b and c. [3]

By complex conjugate root theorem,
$w^* = -1 - 2i$.

$$z^3 + az^2 + bz + c = [z - (-1+2i)][z - (-1-2i)](z-3)$$
$$= (z^2 + z + 2zi + z - 2zi + 5)(z-3)$$
$$= (z^2 + 2z + 5)(z-3)$$
$$= z^3 - z^2 - z - 15$$

By comparison of coefficients:
$a = -1$, $b = -1$ and $c = -15$.

4 **(a)** Without using a calculator, solve the inequality $\frac{4}{x+2} \geqslant \frac{x-3}{x}$. [4]

$$\frac{4}{x+2} - \frac{x-3}{x} \geqslant 0$$

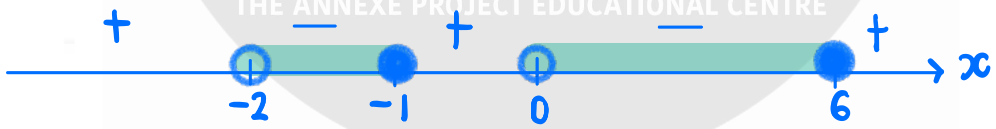
$$\frac{4x-(x-3)(x+2)}{(x+2)(x)} \geqslant 0$$

$$\frac{4x - x^2 + x + 6}{(x+2)(x)} \geqslant 0$$

$$\frac{-x^2+5x+6}{x(x+2)} \geqslant 0$$

$$\frac{x^2-5x-6}{x(x+2)} \leqslant 0$$

$$\frac{(x-6)(x+1)}{x(x+2)} \leqslant 0 \quad , \quad x \neq 0 , x \neq -2$$



$\therefore \quad \underline{-2 < x \leqslant -1 \quad \text{or} \quad 0 < x \leqslant 6}$

(b) Hence, solve the inequality $\frac{4}{|x|+2} \geqslant \frac{|x|-3}{|x|}$. [2]

Replace x with $|x|$:

$-2 < |x| \leqslant -1$ has no solutions.

$0 < |x| \leqslant 6 \longrightarrow \underline{-6 \leqslant x \leqslant 6 \quad , \quad x \neq 0}$

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5 A geometric series has first term a and common ratio r, where $r < 0$. The sum to infinity of the series is 18 and its third term is $\frac{8}{3}$.

(a) Show that $27r^3 - 27r^2 + 4 = 0$. [3]

Given $\frac{a}{1-r} = 18$

$a = 18(1-r)$ —— 1

also, $ar^2 = \frac{8}{3}$ —— 2

Sub 1 into 2: $18(1-r)r^2 = \frac{8}{3}$

$54(r^2 - r^3) = 8$

$54r^3 - 54r^2 + 8 = 0$

$27r^3 - 27r^2 + 4 = 0$ (shown).

(b) Find the values of r and a. [1]

By GC : $r = -\frac{1}{3}$ or $\frac{2}{3}$ (rej. because $r < 0$)

Sub $r = -\frac{1}{3}$ into ① : $a = 18(1 + \frac{1}{3})$

$\therefore a = 24$

The nth term of the series is u_n.

(c) It is given that

$$80 \sum_{r=k+1}^{\infty} u_r = \sum_{r=1}^{k} u_r.$$

Find the value of k. [3]

Given $80 \sum_{r=k+1}^{\infty} u_r = \sum_{r=1}^{k} u_r$

$$80\left[\sum_{r=1}^{\infty} u_r - \sum_{r=1}^{k} u_r\right] = \sum_{r=1}^{k} u_r$$

$$80 \sum_{r=1}^{\infty} u_r = 81 \sum_{r=1}^{k} u_r$$

$$80(18) = 81\left[\frac{24(1-(-\frac{1}{3})^k)}{1-(-\frac{1}{3})}\right]$$

$$1440 = 1458\left[1-(-\tfrac{1}{3})^k\right]$$

$$\frac{80}{81} = 1 - (-\tfrac{1}{3})^k$$

$$(-\tfrac{1}{3})^k = 1 - \frac{80}{81}$$

$$(-\tfrac{1}{3})^k = \frac{1}{81}$$

$\therefore k = 4$

6 A curve C has equation $y = 2e^{x^3} + a$, where a is a positive constant. The point T lies on C and has an x-coordinate of 1.

(a) Use calculus to find the equation of the tangent to C at T. Give the equation in the form $y = e(px + q) + ra$, where p, q and r are exact constants to be found. [5]

When $x = 1$, $y = 2e + a$.

$\therefore T = (1, 2e + a)$

$\frac{dy}{dx} = 2e^{x^3}(3x^2) = 6x^2 e^{x^3}$

When $x = 1$, $\frac{dy}{dx} = 6e$

Equation of tangent: $y - (2e + a) = 6e(x - 1)$

$y = 2e + a + 6ex - 6e$

$= e(6x - 4) + a$

where $p = 6$, $q = -4$, $r = 1$

(b) It is given that the tangent to C at T passes through the origin.

(i) Find the exact value of a. [1]

$y = e(6x - 4) + a$

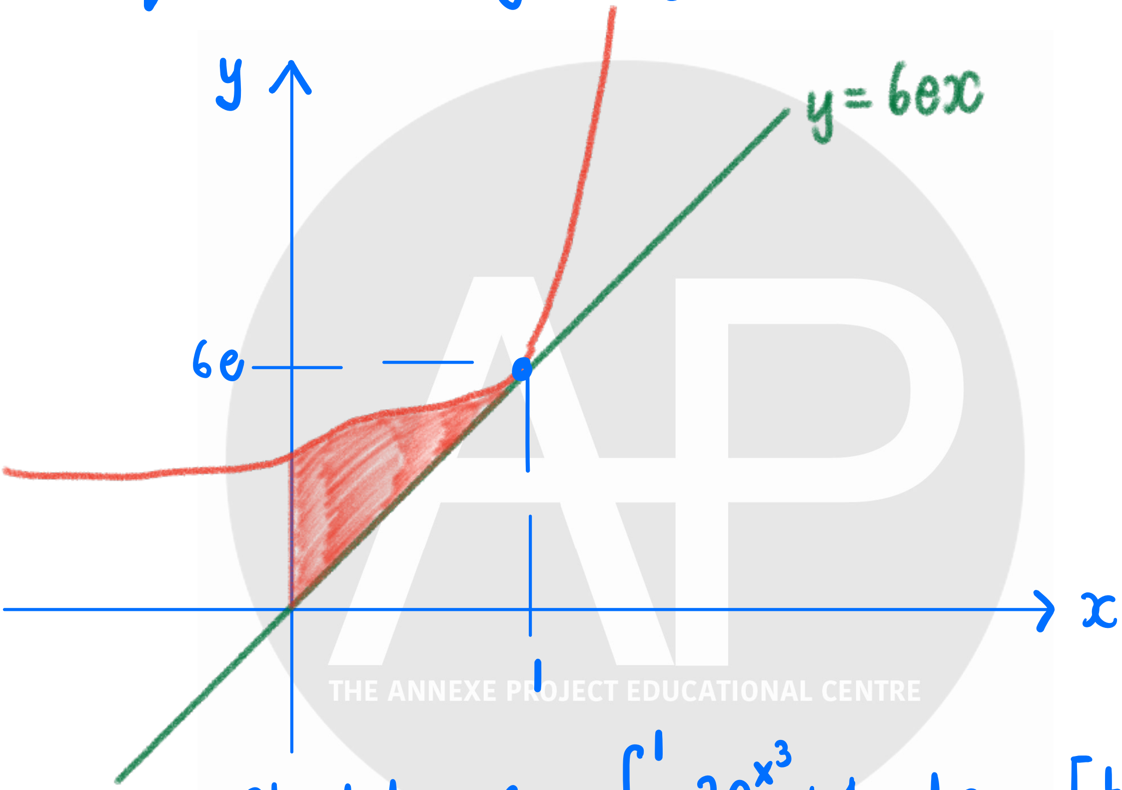
$0 = e(-4) + a$

$\therefore \underline{a = 4e}$

(ii) Find the area of the region bounded by C, the line $x = 0$ and the tangent at T. Give your answer correct to 1 decimal place. [2]

Equation of curve: $y = 2e^{x^3} + 4e$

Equation of tangent: $y = 6ex$



Shaded area $= \int_0^1 2e^{x^3} + 4e \, dx - \left[\frac{1}{2} \times 1 \times 6e\right]$

$= 5.40209$

$= \underline{5.4 \text{ sq units}}$

7 It is given that $f(r) = \cos(r\theta)$.

(a) Show that $f(2r-1) - f(2r+1) = 2\sin\theta\sin(2r\theta)$. [2]

Use Addition Formula

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\cos(2r-1)\theta - \cos(2r+1)\theta$

$= [\cos 2r\theta\cos\theta + \sin 2r\theta\sin\theta] - [\cos 2r\theta\cos\theta - \sin 2r\theta\sin\theta]$

$= \cancel{\cos 2r\theta\cos\theta} + \sin 2r\theta\sin\theta - \cancel{\cos 2r\theta\cos\theta} + \sin 2r\theta\sin\theta$

$= 2\sin\theta\sin 2r\theta$ (shown).

(b) Hence, given that $\sin\theta \neq 0$, show that

$$\sum_{r=1}^{n}\sin(2r\theta) = \frac{\cos\theta - \cos\big((2n+1)\theta\big)}{2\sin\theta}.$$

[3]

$$\sum_{r=1}^{n} 2\sin\theta\sin 2r\theta = \sum_{r=1}^{n} f(2r-1) - f(2r+1)$$

$$2\sin\theta\sum_{r=1}^{n}\sin 2r\theta = f(1) - \cancel{f(3)}$$
$$+ \cancel{f(3)} - \cancel{f(5)}$$
$$+ \cancel{f(5)} - \cancel{f(7)}$$
$$+ \vdots$$
$$+ \cancel{f(2n-1)} - f(2n+1)$$

$$= f(1) - f(2n+1)$$

$$= \cos\theta - \cos(2n+1)\theta$$

$$\therefore \sum_{r=1}^{n}\sin(2r\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta} \quad \text{(shown).}$$

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(c) Hence find the three possible values of

$$\sum_{r=1}^{n} \sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right).$$

[3]

$$\sum_{r=1}^{n} \sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) = \frac{1}{2}\sum_{r=1}^{n} 2\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right)$$

$$= \frac{1}{2}\sum_{r=1}^{n} \sin 2r\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left[\frac{\cos\frac{\pi}{6} - \cos(2n+1)\left(\frac{\pi}{6}\right)}{2\sin\frac{\pi}{6}}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos(2n+1)\left(\frac{\pi}{6}\right)\right]$$

When $n=1$, $\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\frac{\pi}{2}\right] = \frac{\sqrt{3}}{4}$

When $n=2$, $\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\frac{5\pi}{6}\right] = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2}$

When $n=3$, $\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\frac{7\pi}{6}\right] = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2}$

When $n=4$, $\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\frac{9\pi}{6}\right] = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right)$

$= \frac{\sqrt{3}}{4}$

When $n=5$, $\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\frac{11\pi}{6}\right] = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$

$= 0.$

$\therefore$ three possible values are $\underline{0, \frac{\sqrt{3}}{4} \text{ and } \frac{\sqrt{3}}{2}}$

8 It is given that $y = \cos(1 - e^{2x})$.

(a) Show that $\frac{d^2y}{dx^2} = k\left(\frac{dy}{dx} - 2ye^{4x}\right)$, where k is a constant to be found. [3]

$$y = \cos(1 - e^{2x})$$

$$\frac{dy}{dx} = -\sin(1 - e^{2x}) \cdot (-2e^{2x})$$

$$= 2e^{2x}\sin(1 - e^{2x})$$

$$\frac{d^2y}{dx^2} = 2e^{2x} \cdot \cos(1 - e^{2x}) \cdot (-2e^{2x}) + \sin(1 - e^{2x}) \cdot 4e^{2x}$$

$$= -4e^{4x}\cos(1 - e^{2x}) + 2\left[2e^{2x}\sin(1 - e^{2x})\right]$$

$$= -4e^{4x}y + 2\frac{dy}{dx}$$

$$= 2\left(\frac{dy}{dx} - 2ye^{4x}\right) \quad \text{(shown).}$$

(b) By differentiation of the result in part **(a)**, find the first three non-zero terms of the Maclaurin expansion of $\cos(1 - e^{2x})$. [4]

$$\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} - 4y(4e^{4x}) + e^{4x}\left(-4\frac{dy}{dx}\right)$$

$$= 2\frac{d^2y}{dx^2} - 16ye^{4x} - 4\frac{dy}{dx}e^{4x}.$$

$$f(0) = 1$$

$$f'(0) = 2(1)\sin 0 = 0$$

$$f''(0) = 2(0 - 2) = -4$$

$$f'''(0) = 2(-4) - 16 - 4(0) = -24$$

$$\therefore \cos(1 - e^{2x}) = 1 + \frac{x^2}{2!}(-4) + \frac{x^3}{3!}(-24) + \cdots$$

$$= \underline{1 - 2x^2 - 4x^3 + \cdots}$$

(c) The first two non-zero terms of the Maclaurin expansion of $\cos(1-e^{2x})$ are equal to the first two non-zero terms of the series expansion of $\dfrac{1}{\sqrt{a+bx^2}}$, where a and b are constants.

Using standard series from the List of Formulae (MF26), find the values of a and b. [2]

$$\frac{1}{\sqrt{a+bx^2}} = (a+bx^2)^{-\frac{1}{2}}$$
$$= \left[a\left(1+\frac{b}{a}x^2\right)\right]^{-\frac{1}{2}}$$
$$= a^{-\frac{1}{2}}\left[1-\frac{1}{2}\left(\frac{b}{a}x^2\right)+\cdots\right]$$
$$= \frac{1}{\sqrt{a}} - \frac{1}{2\sqrt{a}}\left(\frac{b}{a}\right)x^2+\cdots$$

comparing with $1 - 2x^2 + \cdots$

$\therefore \underline{a=1}$,

$$-\frac{1}{2}\left(\frac{b}{1}\right) = -2$$
$$\frac{b}{2} = 2$$
$$\underline{b=4}$$

9 A curve C has parametric equations

$$x = 3t^2 + 2t, \quad y = t^2 + 2t^3 \quad \text{for } t \geqslant 0.$$

(a) Show that $\frac{dy}{dx} = kt$, where k is a constant to be found. [2]

$$\frac{dx}{dt} = 6t + 2$$

$$\frac{dy}{dt} = 2t + 6t^2$$

$$\therefore \frac{dy}{dx} = \frac{2t + 6t^2}{6t + 2} = \frac{2t\cancel{(1+3t)}}{2\cancel{(1+3t)}} = t \quad \text{where } \underline{k = 1}$$

(b) The tangent to C at a point P makes an angle of $60°$ with the x-axis. Find the exact coordinates of P. [3]

$$\text{gradient} = \tan 60° = \sqrt{3}$$

$$\text{Let } t = \sqrt{3}$$

$$\therefore P = \left(3\sqrt{3}^2 + 2\sqrt{3}, \sqrt{3}^2 + 2\sqrt{3}^3\right)$$

$$= \underline{\left(9 + 2\sqrt{3}, 3 + 6\sqrt{3}\right)}$$

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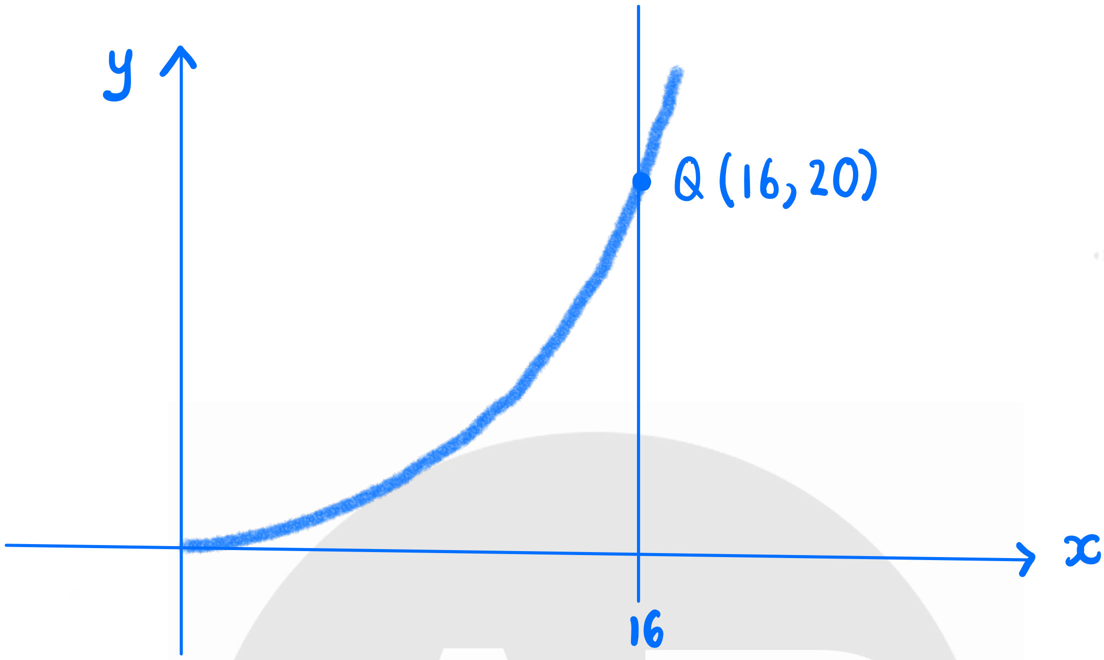
The point Q with coordinates (16, 20) lies on C.

(c) Using calculus, find the exact area of the region bounded by the curve C, the x-axis and the line parallel to the y-axis through Q. [5]



When $x = 16$, $3t^2 + 2t = 16$

$$3t^2 + 2t - 16 = 0$$

$$(3t+8)(t-2) = 0$$

$t = -\frac{8}{3}$ (rej.) or $t = 2$

$$\text{Area} = \int_{x=0}^{x=16} y \, dx = \int_{t=0}^{t=2} (t^2+2t^3)(6t+2)\, dt$$

$$= \int_0^2 12t^4 + 10t^3 + 2t^2 \, dt$$

$$= \left[\frac{12t^5}{5} + \frac{5t^4}{2} + \frac{2t^3}{3} \right]_0^2$$

$$= \frac{384}{5} + 40 + \frac{16}{3}$$

$$= \underline{\frac{1832}{15} \text{ sq units.}}$$

10 **(a)** Using the substitution $u = \sqrt{4x+1}$, show that $\displaystyle\int \frac{\sqrt{4x+1}}{x-2}\,dx = \int \frac{2u^2}{u^2-9}\,du$. [3]

Step 1: $\dfrac{du}{dx} = \frac{1}{2}(4x+1)^{-\frac{1}{2}}(4)$ also $u = \sqrt{4x+1}$

$= \dfrac{2}{\sqrt{4x+1}}$ $u^2 = 4x+1$

$= \dfrac{2}{u}$ $x = \dfrac{u^2-1}{4}$

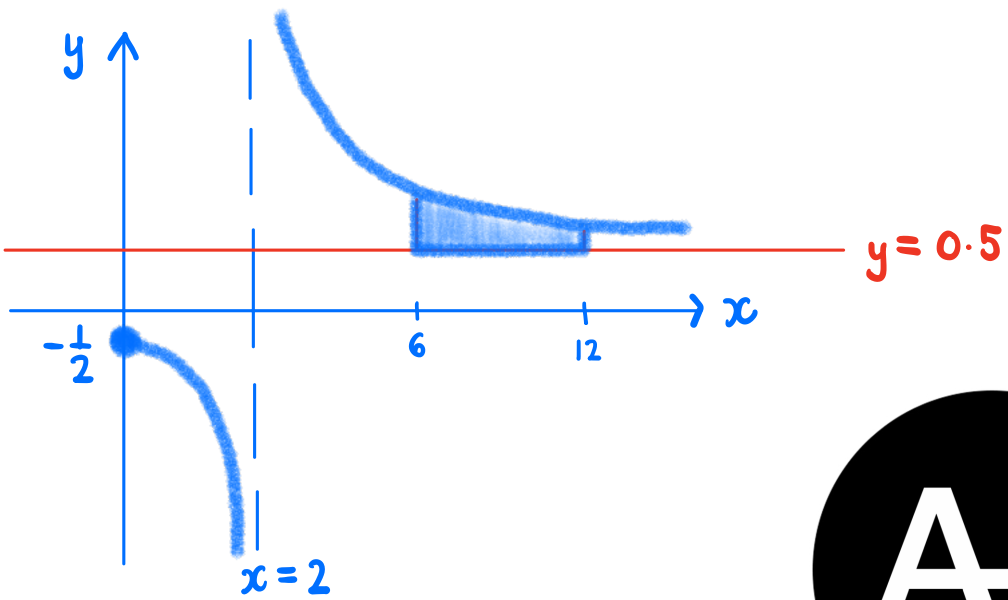
Step 2: $\displaystyle\int \frac{\sqrt{4x+1}}{x-2}\,dx = \int \frac{u}{\frac{u^2-1}{4}-2} \cdot \frac{u}{2}\,du$

$\displaystyle= \int \frac{\overset{2}{\cancel{4}}u}{u^2-1-8} \cdot \frac{u}{\cancel{2}}\,du$

$\displaystyle= \int \frac{2u^2}{u^2-9}\,du$ (shown).

(b) The region R lies in the first quadrant and is bounded by the curve $y = \dfrac{\sqrt{4x+1}}{x-2}$ and the lines $x = 6$, $x = 12$ and $y = 0.5$.

(i) Sketch the graph of $y = \dfrac{\sqrt{4x+1}}{x-2}$ for $x \geqslant 0$, giving the coordinates of any intercepts with the axes and the equations of any asymptotes. Shade the region R on your sketch. [3]



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(ii) Find the exact area of R, giving your answer in the form $a+b\ln c$. [5]

when $x=12$, $u=\sqrt{48+1}=7$
when $x=6$, $u=\sqrt{24+1}=5$

$$\int_{x=6}^{x=12}\frac{\sqrt{4x+1}}{x-2}\,dx=\int_5^7\frac{2u^2}{u^2-9}\,du$$

Apply Long Division

$$\begin{array}{r} 2 \\ u^2-9\,\big)\overline{2u^2} \\ -(2u^2-18) \\ \hline 18 \end{array}$$

$$=\int_5^7 2+\frac{18}{u^2-9}\,du$$

$$=\left[2u\right]_5^7+18\int_5^7\frac{1}{u^2-3^2}\,du$$

$$=(14-10)+18\cdot\frac{1}{6}\left[\ln\left|\frac{u-3}{u+3}\right|\right]_5^7$$

$$=4+3\left(\ln\frac{2}{5}-\ln\frac{1}{4}\right)$$

$$=4+3\ln\frac{8}{5}$$

$\therefore$ Exact area of $R=\left[4+3\ln\frac{8}{5}\right]-(6\times0.5)$
$=\left(3\ln\frac{8}{5}\right)+1$ sq units.

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis. Give your answer correct to 2 decimal places. [3]

$$\pi\int_6^{12}\left(\frac{\sqrt{4x+1}}{x-2}\right)^2dx-\underbrace{\left[\pi(0.5)^2(6)\right]}_{\text{volume of cylinder}}$$

$=11.04321$
$=11.04$ cubic units.

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11 A helicopter is hovering at a great height above the ground. A parachutist drops from the helicopter and his parachute opens. The parachutist falls vertically. At time t seconds after leaving the helicopter he has fallen a distance of x metres. At this time he has velocity $v\,\mathrm{m\,s^{-1}}$. The motion of the parachutist is modelled by the differential equation

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}=9.8,$$

where α is a constant.

(a) **(i)** Use the equation $v=\frac{dx}{dt}$ to write down a differential equation in v and t. [1]

Since $\frac{dv}{dt}=\frac{d^2x}{dt^2}$, from the above given d.e:

$\frac{dv}{dt}+\alpha v=9.8$

(ii) Given that $\frac{dv}{dt}=5$ when $v=2.4$, find the value of α. [1]

$5+\alpha(2.4)=9.8$

$2.4\alpha=4.8$

$\alpha=2$

(b) It is given that the initial velocity of the parachutist is zero. Show that $v=A(1-e^{Bt})$, where A and B are constants to be found. [5]

$\frac{dv}{dt}+2v=9.8$

$\frac{dv}{dt}=9.8-2v$

$\int\frac{1}{9.8-2v}\,dv=\int dt$

$-\frac{1}{2}\int\frac{-2}{9.8-2v}\,dv=t+C$

$-\frac{1}{2}\ln|9.8-2v|=t+C$

$\ln|9.8-2v|=-2t-2C$

$|9.8-2v|=e^{-2t-2C}$

$9.8-2v=\pm e^{-2C}e^{-2t}$

$9.8-2v=Ke^{-2t}$ where $K=\pm e^{-2C}$

$2v=9.8-Ke^{-2t}$

$v=4.9-\frac{K}{2}e^{-2t}$

When $t=0$, $v=0$: $0=4.9-\frac{K}{2}$

$\therefore\ \frac{K}{2}=4.9$

Hence, $v=4.9-4.9e^{-2t}=4.9(1-e^{-2t})$

where $A=4.9$, $B=-2$

(c) The helicopter is at a height of 2000 m above the ground. Find the time taken for the parachutist to reach the ground. [5]

$v = 4.9(1-e^{-2t})$

$\therefore \frac{dx}{dt} = 4.9(1-e^{-2t})$

$x = 4.9\int 1-e^{-2t}\,dt$

$= 4.9\left[t - \frac{e^{-2t}}{-2}\right] + c$

$= 4.9t + 2.45e^{-2t} + c$

When $t=0$, $x=0$: $c = -2.45$

$\therefore x = 4.9t + 2.45e^{-2t} - 2.45$

Let $x = 2000$: By GC: $t = 408.66327$

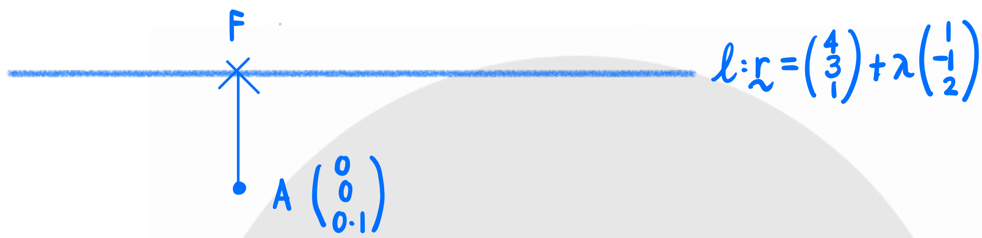
$= 409$ s

12 Air traffic controllers need to know the exact locations of aircraft. They use coordinates (x, y, z), with units in kilometres, to locate individual aircraft relative to the base of the control tower which is at position $(0, 0, 0)$. The ground is horizontal and is modelled as the x–y plane.

An aircraft flies along a path with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. The air traffic controllers are situated in the control tower 0.1 km above the base of the control tower.

(a) Due to the atmospheric conditions at the time, the air traffic controllers can only see aircraft that come within 4 km of their position.

(i) Find the coordinates of this aircraft when it is at its closest point to the air traffic controllers. [4]



Let F be the point of the aircraft where its closest to the air traffic controllers A.

Step 1 : $\vec{OF} = \begin{pmatrix} 4+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix}$

Step 2 : $\vec{AF} = \begin{pmatrix} 4+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ 3-\lambda \\ 0.9+2\lambda \end{pmatrix}$

Step 3 $\vec{AF} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$

$4+\lambda-3+\lambda+1.8+4\lambda = 0$

$\lambda = -\frac{7}{15}$

Hence $\vec{OF} = \begin{pmatrix} 53/15 \\ 52/15 \\ 1/15 \end{pmatrix}$ $\quad \therefore F = \left(\frac{53}{15}, \frac{52}{15}, \frac{1}{15}\right)$

(ii) Hence determine whether the air traffic controllers will see this aircraft. [2]

$|\vec{AF}| = \left|\begin{pmatrix} 53/15 \\ 52/15 \\ -1/30 \end{pmatrix}\right| = \sqrt{\left(\frac{53}{15}\right)^2 + \left(\frac{52}{15}\right)^2 + \left(\frac{-1}{30}\right)^2}$

$= 4.95 > 4$

The aircraft cannot be seen.

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A drone flies along a path with equation

$$\frac{2x-3}{3} = y+1 = \frac{z-k}{-1}, \quad = \mu$$

where k is a positive constant.

(b) Determine the possible values of k if the paths of the aircraft and the drone do not intersect. [3]

$$l: \underset{\sim}{r} = \begin{pmatrix} 1.5 \\ -1 \\ k \end{pmatrix} + \mu \begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix}, \ \mu \in \mathbb{R}$$

Since the two paths are not parallel, they must be skew.

$$\text{Let } \begin{pmatrix} 4+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} 1.5+1.5\mu \\ -1+\mu \\ k-\mu \end{pmatrix}$$

Solving

$$4+\lambda = 1.5+1.5\mu$$

$$\lambda = -2.5+1.5\mu \quad \text{—— (1)}$$

$$3-\lambda = -1+\mu$$

$$\lambda = 4-\mu \quad \text{—— (2)}$$

$$\mu = 2.6, \quad \lambda = 1.4$$

Let $1+2\lambda = k-\mu$

$$k = 1+2\lambda+\mu$$

$$= 1+2(1.4)+2.6$$

$$= 6.4$$

Hence, $k \in \mathbb{R} \setminus \{6.4\}$

(c) Find the acute angle between the paths of the aircraft and the drone. [2]

$$\theta = \cos^{-1}\left|\frac{\begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{4.25}\sqrt{6}}\right|$$

$$= \cos^{-1}\left|\frac{-1.5}{\sqrt{4.25}\sqrt{6}}\right|$$

$$= 72.7^\circ$$

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12 [Continued]

(d) It is now given that $k = 2$.

The aircraft and the drone are observed on radar to be closest to each other when they are at the points (4, 3, 1) and (4.2, 0.8, 0.2) on their respective paths.

(i) Find the distance between the aircraft and the drone at this instant. [2]

Let $\overrightarrow{OP} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 4.2 \\ 0.8 \\ 0.2 \end{pmatrix}$

then $\overrightarrow{PQ} = \begin{pmatrix} 4.2 \\ 0.8 \\ 0.2 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -2.2 \\ -0.8 \end{pmatrix}$

$$|\overrightarrow{PQ}| = \sqrt{0.2^2 + (-2.2)^2 + (-0.8)^2}$$
$$= \underline{2.35 \text{ km}}.$$

(ii) By considering the vector between the aircraft and the drone at this instant, determine whether the distance found in part **(d)(i)** is the shortest distance between the two paths. [2]

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -2.2 \\ -0.8 \end{pmatrix} \cdot \begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix}$$
$$= -1.1 \neq 0$$

Since $\overrightarrow{PQ}$ is not perpendicular to the direction of the drone's path, $|\overrightarrow{PQ}|$ is <u>not</u> the shortest distance between the two paths.

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