MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS 9758/02

Paper 2

October/November 2018

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

[Turn over

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(2018)5
H2 Mathematics

Section A: Pure Mathematics [40 marks]

1 The curve $y = f(x)$ passes through the point $(0, 69)$ and has gradient given by

$$\frac{dy}{dx} = \left(\tfrac{1}{3}y - 15\right)^{\frac{1}{3}}.$$

(i) Find $f(x)$. [4]

(ii) Find the coordinates of the point on the curve where the gradient is 4. [2]

(i). $\frac{dy}{dx} = \left(\frac{1}{3}y - 15\right)^{\frac{1}{3}}$

$$\int \left(\tfrac{1}{3}y - 15\right)^{-\frac{1}{3}} dy = \int dx$$

$$\frac{\left(\frac{1}{3}y-15\right)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)} = x + c$$

$$\tfrac{9}{2}\left(\tfrac{1}{3}y - 15\right)^{\frac{2}{3}} = x + C$$

When $x = 0$, $y = 69$: $\frac{9}{2}\left(\frac{1}{3} \times 69 - 15\right)^{\frac{2}{3}} = C$

$\therefore C = 18$

$\therefore \frac{9}{2}\left(\frac{1}{3}y - 15\right)^{\frac{2}{3}} = x + 18$

$\left(\frac{1}{3}y - 15\right)^{\frac{2}{3}} = \frac{2}{9}x + 4$

$\frac{1}{3}y - 15 = \left(\frac{2}{9}x + 4\right)^{\frac{3}{2}}$

$\frac{1}{3}y = 15 + \left(\frac{2}{9}x + 4\right)^{\frac{3}{2}}$

$\underline{y = 45 + 3\left(\frac{2}{9}x + 4\right)^{\frac{3}{2}}}$

(ii). Let $4 = \left(\frac{1}{3}y - 15\right)^{\frac{1}{3}}$

$64 = \frac{1}{3}y - 15$

$\therefore y = 237$

When $y = 237$, $237 = 45 + 3\left(\frac{2}{9}x + 4\right)^{\frac{3}{2}}$

$64 = \left(\frac{2}{9}x + 4\right)^{\frac{3}{2}}$

$16 = \frac{2}{9}x + 4$

$x = 54$

$\underline{\text{coordinates} = (54, 237)}$

2 (a) One of the roots of the equation $4x^4 - 20x^3 + sx^2 - 56x + t = 0$, where s and t are real, is $2 - 3i$. Find the other roots of the equation and the values of s and t. [5]

Since all the coefficients are real,
by complex conjugate root theorem, $x = 2+3i$ is also a root.
$[x-(2-3i)][x-(2+3i)] = (x^2 - 4x + 13)$ is a quadratic factor.

Sub $(2-3i)$ into the equation:

$4(2-3i)^4 - 20(2-3i)^3 + s(2-3i)^2 - 56(2-3i) + t = 0$

$-476 + 480i + 920 + 180i - 5s - 12si - 112 + 168i + t = 0$

$(332 - 5s + t) + (828 - 12s)i = 0$

By comparison of coefficient:

$12s = 828$, $332 - 5s + t = 0$

$s = 69$, $t = 5(69) - 332 = 13$

$4x^4 - 20x^3 + 69x^2$
$= (x^2 - 4x + 13)(4x^2 - 4x + 1)$
$= (x^2 - 4x + 13)(2x - 1)(2x - 1)$

$\therefore x = 2 - 3i,\ 2 + 3i$ or $\frac{1}{2}$

$$
\begin{array}{r}
4x^2 - 4x + 1 \\
x^2 - 4x + 13 \,\big)\, 4x^4 - 20x^3 + 69x^2 - 56x + 13 \\
-(4x^4 - 16x^3 + 52x^2) \\
\hline
-4x^3 + 17x^2 - 56x + 13 \\
-(-4x^3 + 16x^2 - 52x) \\
\hline
x^2 - 4x + 13 \\
-(x^2 - 4x + 13) \\
\hline
\end{array}
$$

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(b) The complex number w is such that $w^3 = 27$.

(i) Given that one possible value of w is 3, use **a non-calculator method** to find the other possible values of w. Give your answers in the form $a + \mathrm{i}b$, where a and b are exact values. [3]

Step 1 :

Let $w^3 - 27 = 0$

and since $(w-3)$ is a factor,

by Long Division ,

$w^3 - 27 = (w-3)(w^2+3w+9)$

$$\begin{array}{r@{}l} & w^2 + 3w + 9 \\ w-3 \,\big) & w^3 + 0w^2 + 0w - 27 \\ & -(w^3 - 3w^2) \\ \hline & 3w^2 + 0w - 27 \\ & -(3w^2 - 9w) \\ \hline & 9w - 27 \\ & -(9w - 27) \\ \hline & 0 \end{array}$$

Step 2 :

Solving for $w^2 + 3w + 9 = 0$

$$w = \frac{-3 \pm \sqrt{9 - 4(1)(9)}}{2}$$

$$= \frac{-3 \pm \sqrt{-27}}{2}$$

$$= \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}\,i}{2}$$

other values of w are $\underline{\frac{-3}{2} - \frac{3\sqrt{3}}{2}i}$ and $\underline{-\frac{3}{2} + \frac{3\sqrt{3}}{2}i}$

(ii) Write these values of w in modulus-argument form and represent them on an Argand diagram. [2]

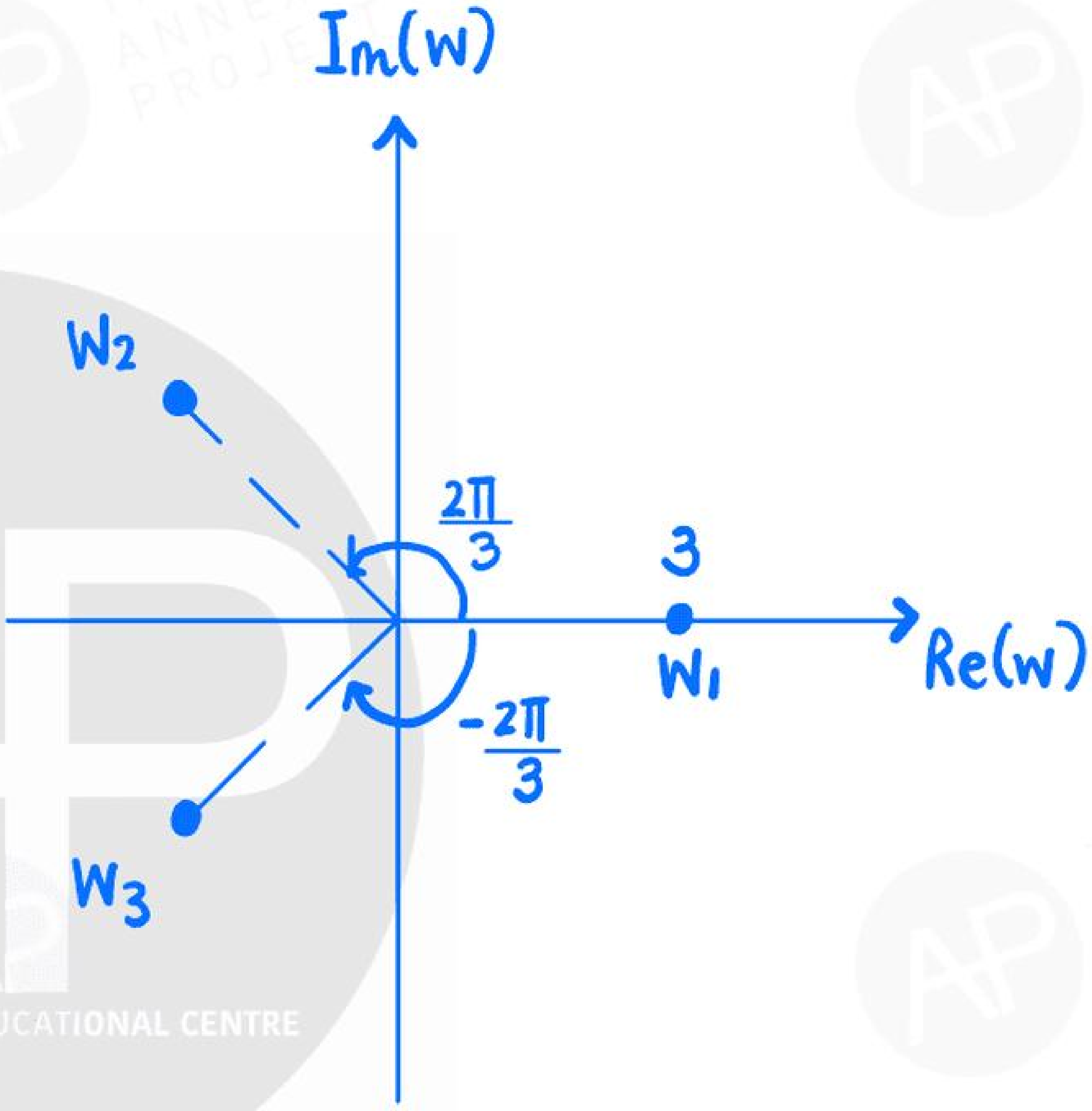
Let $w_1 = 3$

$w_2 = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

$w_3 = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

$|w_1| = 3$, $\arg w_1 = 0$

$\therefore \underline{w_1 = 3[\cos 0 + i\sin 0]}$



$|w_2| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{27}{4}} = 3$

$\arg w_2 = \pi - \tan^{-1}\left(\frac{\frac{3\sqrt{3}}{2}}{\frac{3}{2}}\right)$

$= \pi - \tan^{-1}\sqrt{3}$

$= \frac{2\pi}{3}$

$\therefore \underline{w_2 = 3\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]}$

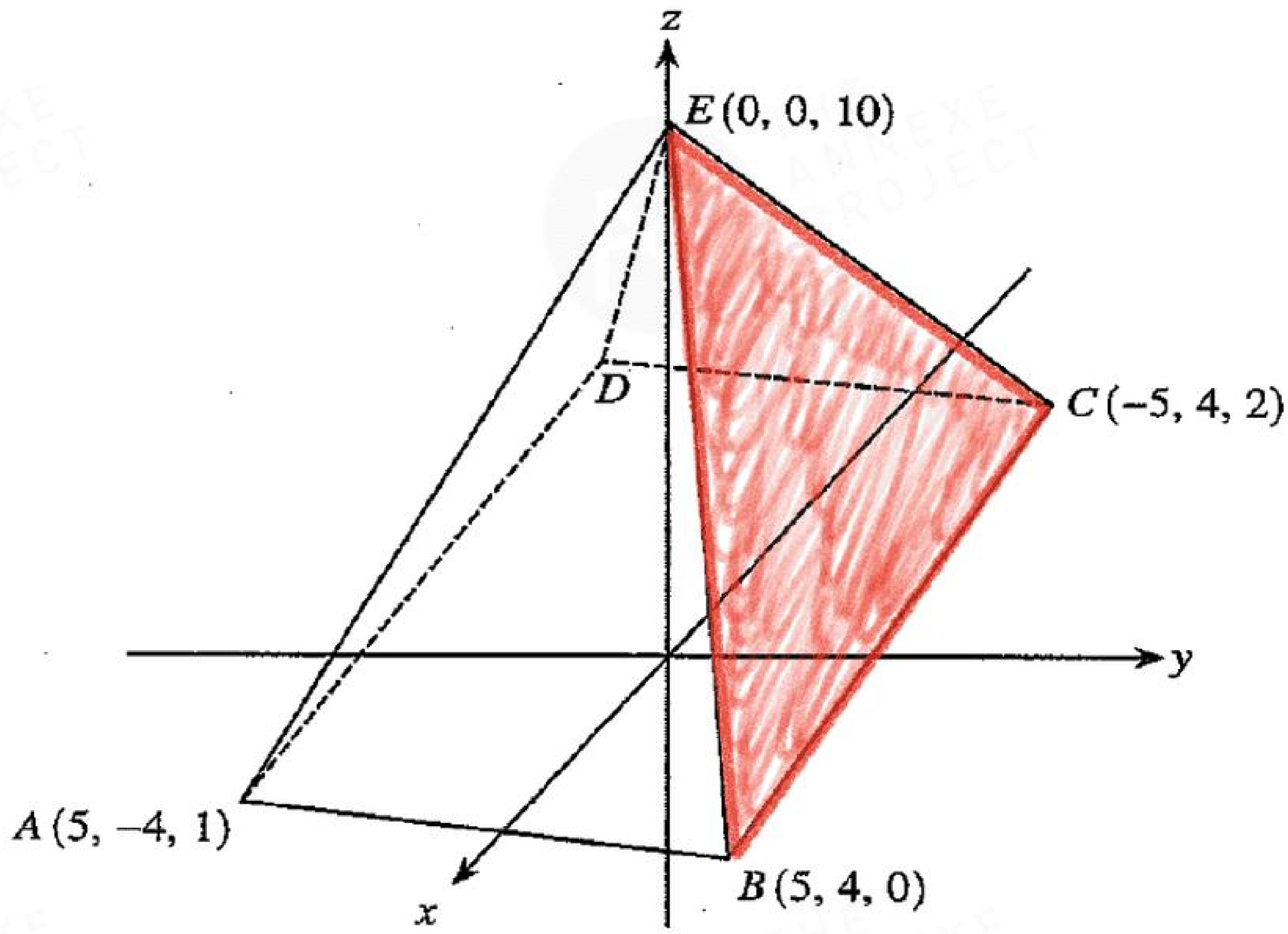
Since w_3 is conjugate of w_2,

$\therefore \underline{w_3 = 3\left[\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right]}$

(iii) Find the sum and the product of all the possible values of w, simplifying your answers. [2]

$w_1 + w_2 + w_3 = 3 + \left(-\frac{3}{2} + \frac{3\sqrt{2}}{2}i\right) + \left(-\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right) = \underline{0}$

$w_1 w_2 w_3 = 3\left(-\frac{3}{2} + \frac{3\sqrt{2}}{2}i\right)\left(-\frac{3}{2} - \frac{3\sqrt{2}}{2}i\right) = \underline{27}$ (by G.C)



An oblique pyramid has a plane base $ABCD$ in the shape of a parallelogram. The coordinates of A, B and C are $(5, -4, 1)$, $(5, 4, 0)$ and $(-5, 4, 2)$ respectively. The apex of the pyramid is at $E\,(0, 0, 10)$ (see diagram).

(i) Find the coordinates of D. [1]

(ii) Find the cartesian equation of face BCE. [3]

(i) $\overrightarrow{AD} = \overrightarrow{BC}$

$$\overrightarrow{OD} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OD} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$$

$\therefore$ coordinates of D = $\underline{(-5, -4, 3)}$

(ii) $\overrightarrow{BC} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$, $\overrightarrow{BE} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix}$

$$\underset{\sim}{n} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix} = \begin{pmatrix} 0-(-8) \\ -10-(-100) \\ 40-0 \end{pmatrix} = \begin{pmatrix} 8 \\ 90 \\ 40 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}$$

$$\pi: \underset{\sim}{r} \cdot \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}$$
$$= 200$$

$\therefore$ $\underline{4x + 45y + 20z = 200}$

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(iii) Find the angle between face BCE and the base of the pyramid. [3]

(iv) Find the shortest distance from the midpoint of edge AD to face BCE. [5]

(iii)

$$\underset{\sim}{n}_2 = \overrightarrow{AB} \times \overrightarrow{BC} = \left[\begin{pmatrix}5\\4\\0\end{pmatrix} - \begin{pmatrix}5\\-4\\1\end{pmatrix}\right] \times \begin{pmatrix}-10\\0\\2\end{pmatrix} = \begin{pmatrix}0\\8\\-1\end{pmatrix} \times \begin{pmatrix}-10\\0\\2\end{pmatrix}$$

$$= \begin{pmatrix}16-0\\10-0\\0-(-80)\end{pmatrix} = \begin{pmatrix}16\\10\\80\end{pmatrix} = 2\begin{pmatrix}8\\5\\40\end{pmatrix}$$

$$\cos^{-1} \frac{\begin{pmatrix}4\\45\\20\end{pmatrix} \cdot \begin{pmatrix}8\\5\\40\end{pmatrix}}{\sqrt{4^2+45^2+20^2}\sqrt{8^2+5^2+40^2}} = \cos^{-1}\left(\frac{32+225+800}{\sqrt{2441}\sqrt{1689}}\right)$$

$$= 58.630$$

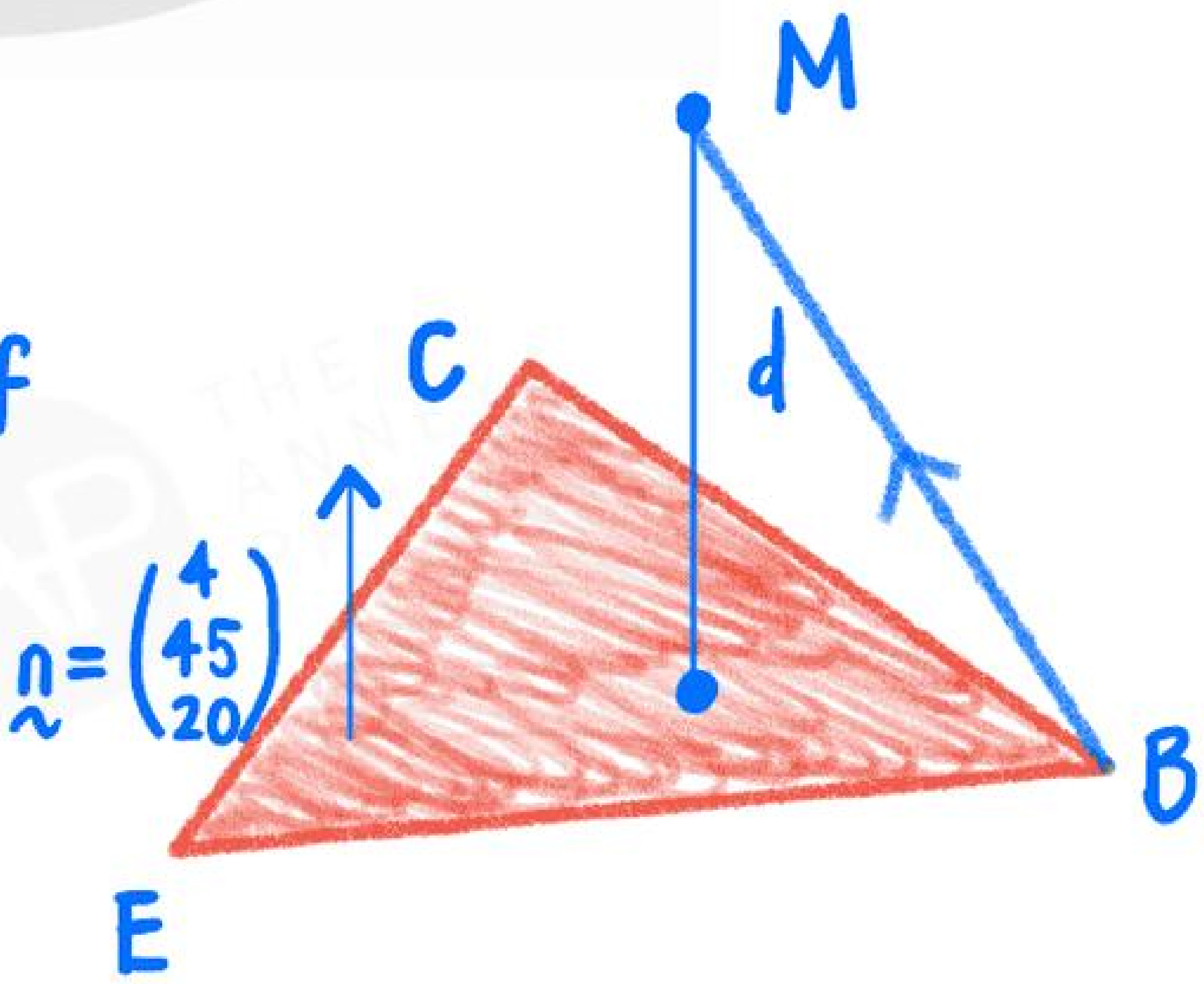
$$= \underline{58.6^\circ}$$

(iv) Step 1:

$$\text{midpoint of } AD = \frac{\begin{pmatrix}5\\-4\\1\end{pmatrix} + \begin{pmatrix}-5\\-4\\3\end{pmatrix}}{2}$$

$$\overrightarrow{OM} = \begin{pmatrix}0\\-4\\2\end{pmatrix}$$

Let the shortest distance of M to plane BCE be d.



Step 2:

$$d = \left|\overrightarrow{BM} \cdot \hat{\underset{\sim}{n}}\right|$$

$$= \left|\frac{\left\{\begin{pmatrix}0\\-4\\2\end{pmatrix} - \begin{pmatrix}5\\4\\0\end{pmatrix}\right\} \cdot \begin{pmatrix}4\\45\\20\end{pmatrix}}{\sqrt{4^2+45^2+20^2}}\right| = \left|\frac{\begin{pmatrix}-5\\-8\\2\end{pmatrix} \cdot \begin{pmatrix}4\\45\\20\end{pmatrix}}{\sqrt{2441}}\right| = \left|\frac{-20-360+40}{\sqrt{2441}}\right|$$

$$= \underline{6.88 \text{ units}}$$

4 In this question you may use expansions from the List of Formulae (MF26).

(i) Find the Maclaurin expansion of $\ln(\cos 2x)$ in ascending powers of x, up to and including the term in x^6. State any value(s) of x in the domain $0 \leqslant x \leqslant \frac{1}{4}\pi$ for which the expansion is **not** valid. [6]

(ii) Use your expansion from part (i) and integration to find an approximate expression for $\int \frac{\ln(\cos 2x)}{x^2}\,dx$. Hence find an approximate value for $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx$, giving your answer to 4 decimal places. [3]

(iii) Use your graphing calculator to find a second approximate value for $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx$, giving your answer to 4 decimal places. [1]

(i)
$$\ln(\cos 2x) = \ln\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$
$$= \ln\left[1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720} + \cdots\right]$$
$$= \ln\left[1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right]$$
$$\approx \ln\left[1 + \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)\right]$$
$$= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) - \frac{\left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)^2}{2} + \frac{\left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)^3}{3} - \cdots$$
$$= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) - \frac{1}{2}\left(4x^4 - \frac{8}{3}x^6 + \cdots\right) + \frac{1}{3}\left(-8x^6 + \cdots\right)$$
$$= \underline{-2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6 + \cdots}$$

When $x = \frac{\pi}{4}$, $\ln\left(\cos 2\left(\frac{\pi}{4}\right)\right) = \ln 0$ (undefined).

$\therefore$ when $x = \frac{\pi}{4}$, the expansion is not valid.

(ii)
$$\int \frac{\ln(\cos 2x)}{x^2}\,dx \approx \int \frac{-2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6}{x^2}\,dx$$
$$= \int -2 - \frac{4}{3}x^2 - \frac{64}{45}x^4\,dx$$
$$= -2x - \frac{4}{9}x^3 - \frac{64}{225}x^5 + C$$

$$\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx = \left[-2x - \frac{4}{9}x^3 - \frac{64}{225}x^5\right]_0^{0.5}$$

$$= -1.0644 \text{ (4 d.p.)}$$

(iii). By GC: $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx = -1.0670$ (4 d.p.)

Section B: Probability and Statistics [60 marks]

5 The manufacturer of a certain type of fan used for cooling electronic devices claims that the mean time to failure (MTTF) is 65 000 hours. The quality control manager suspects that the MTTF is actually less than 65 000 hours and decides to carry out a hypothesis test on a sample of these fans. (An accelerated testing procedure is used to find the MTTF.)

(i) Explain why the manager should take a sample of at least 30 fans, and state how these fans should be chosen. [2]

(ii) State suitable hypotheses for the test, defining any symbols that you use. [2]

The quality control manager takes a suitable sample of 43 fans, and finds that they have an MTTF of 64 230 hours.

(iii) Given that the manager concludes that there is no reason to reject the null hypothesis at the 5% level of significance, find the range of possible values of the variance used in calculating the test statistic. [3]

(i) Sample size should be at least 30 so that Central Limit Theorem (CLT) can be applied to approximate the distribution to be Normal, and then z-test can be used during the hypothesis testing.

(ii) To test $H_0 : \mu = 65000$ against
$H_1 : \mu < 65000$
where μ represents the population MTTF in hours,
H_0 : null hypothesis
H_1 : alternate hypothesis.

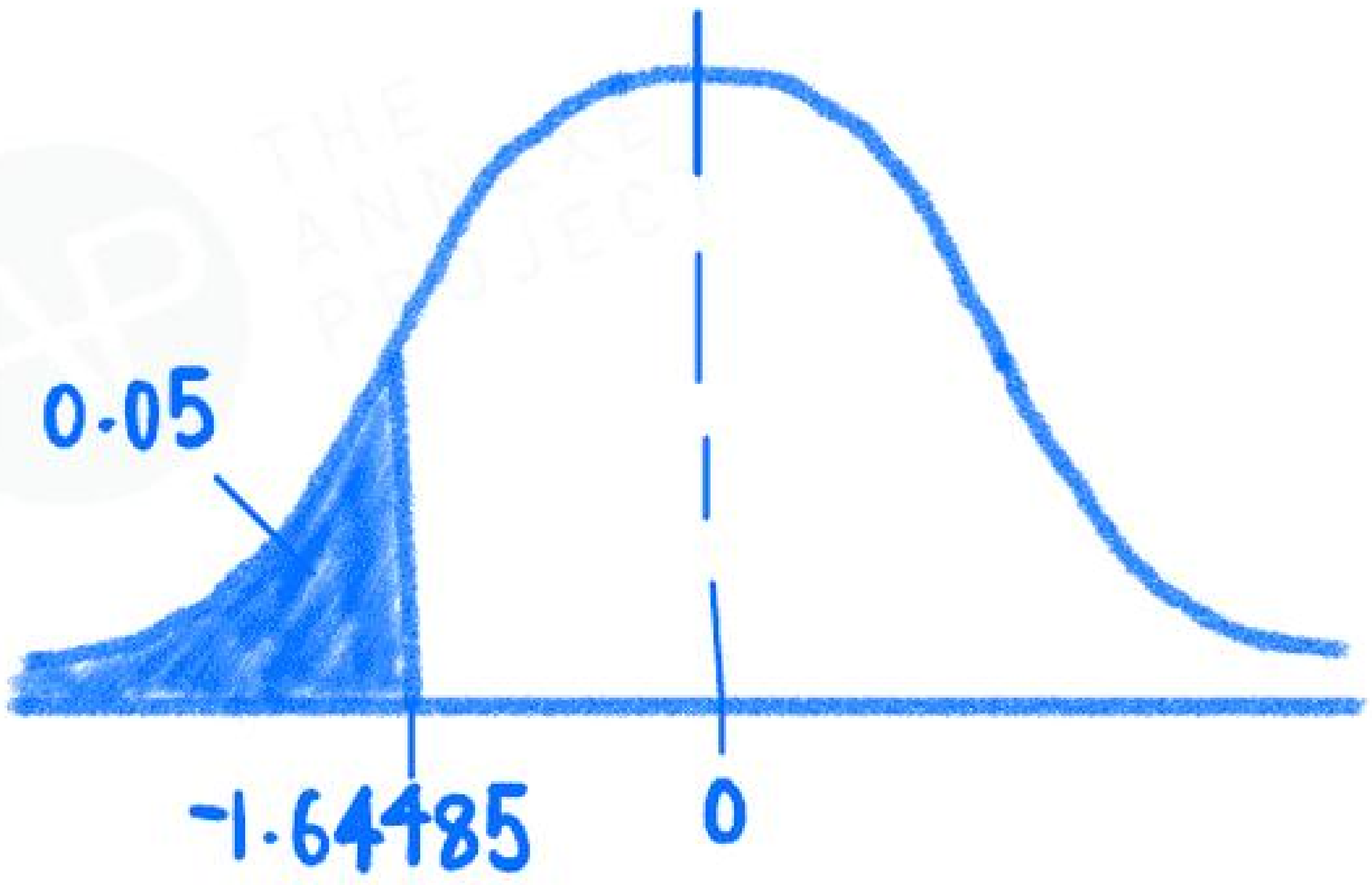
(iii) At 5% level of significance,

since $n=43$ is large, by CLT : $\bar{X} \sim N\left(65000, \frac{\sigma^2}{43}\right)$ approximately

Test Statistics under H_0, $Z = \frac{\bar{X}-\mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$

$$= \frac{64230 - 65000}{\frac{\sigma}{\sqrt{43}}}$$

$$= \frac{\sqrt{43}(-770)}{\sigma}$$



Since H_0 is not rejected,

$z\text{-value} > -1.64485$

$\dfrac{-770\sqrt{43}}{\sigma} > -1.64485$

$-770\sqrt{43} > -1.64485\sigma$

$\therefore \quad \sigma > 3069.719$

$\sigma^2 > 9423176.107$

$\underline{\sigma^2 > 9\,420\,000}$

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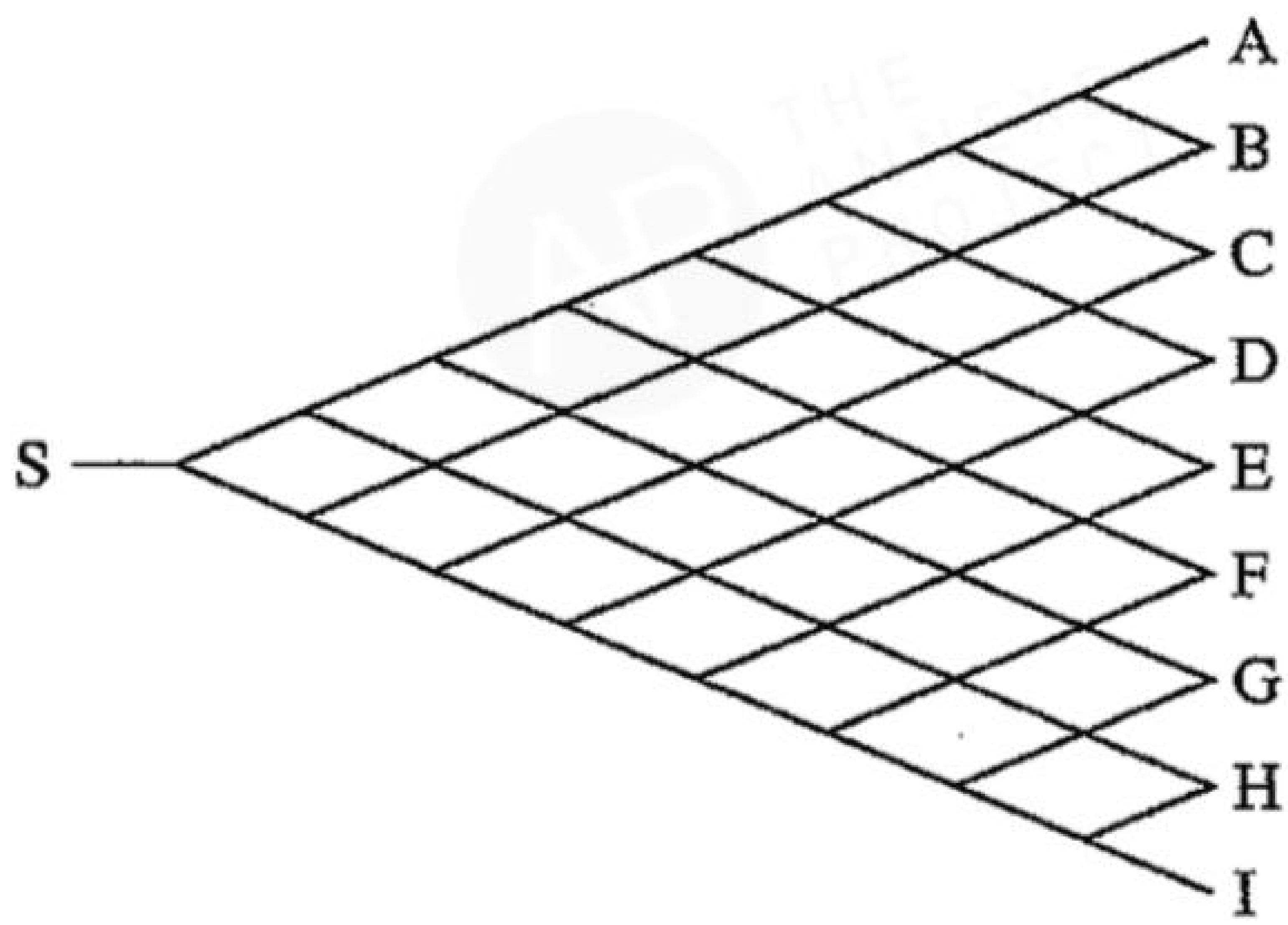


In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A – I (see diagram).

(i) Show that the probability that the bug finishes its journey at D is $56p^5q^3$. [2]

(ii) Given that the probability that the bug finishes its journey at D is greater than the probability that the bug finishes its journey at any one of the other endpoints, find exactly the possible range of values of p. [4]

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(iii) Find the probability that, in this version of the game, the bug reaches one of the endpoints A – I, without being swallowed up by a black hole. [1]

(i). Let X be the r.v. denoting the number of left forks the bug takes out of 8.

$$X \sim B(8, p)$$

$$P(X=5) = \binom{8}{5} p^5 (1-p)^3$$
$$= \underline{56p^5(1-p)^3}$$

(ii). Given that $P(X=5)$ is the largest, it means 5 is the mode of X.

<u>Step 1</u> : $P(X=5) > P(X=4)$

$$56p^5(1-p)^3 > 70p^4(1-p)^4$$
$$4p > 5(1-p)$$

$$9p > 5$$

$$p > \frac{5}{9} \quad \text{———} \ (1)$$

Step 2 : $P(X=5) > P(X=6)$

$$56p^5(1-p)^3 > 28p^6(1-p)^2$$

$$2(1-p) > p$$

$$3p < 2$$

$$p < \frac{2}{3} \quad \text{———} \ (2)$$

Hence, $\underline{\frac{5}{9} < p < \frac{2}{3}}$

(iii) Each Junction
- $0.9p$ — left
- 0.1 — swallowed by black hole
- $0.9q$ — Right

$$0.9p + 0.9q = 0.9(p+q) = 0.9(1) = \underline{0.9}$$

$\therefore$ Each Junction
- 0.1 — swallowed by hole
- 0.9 — not swallowed.

Required Prob. $= 0.9^8$

$= \underline{0.430}$

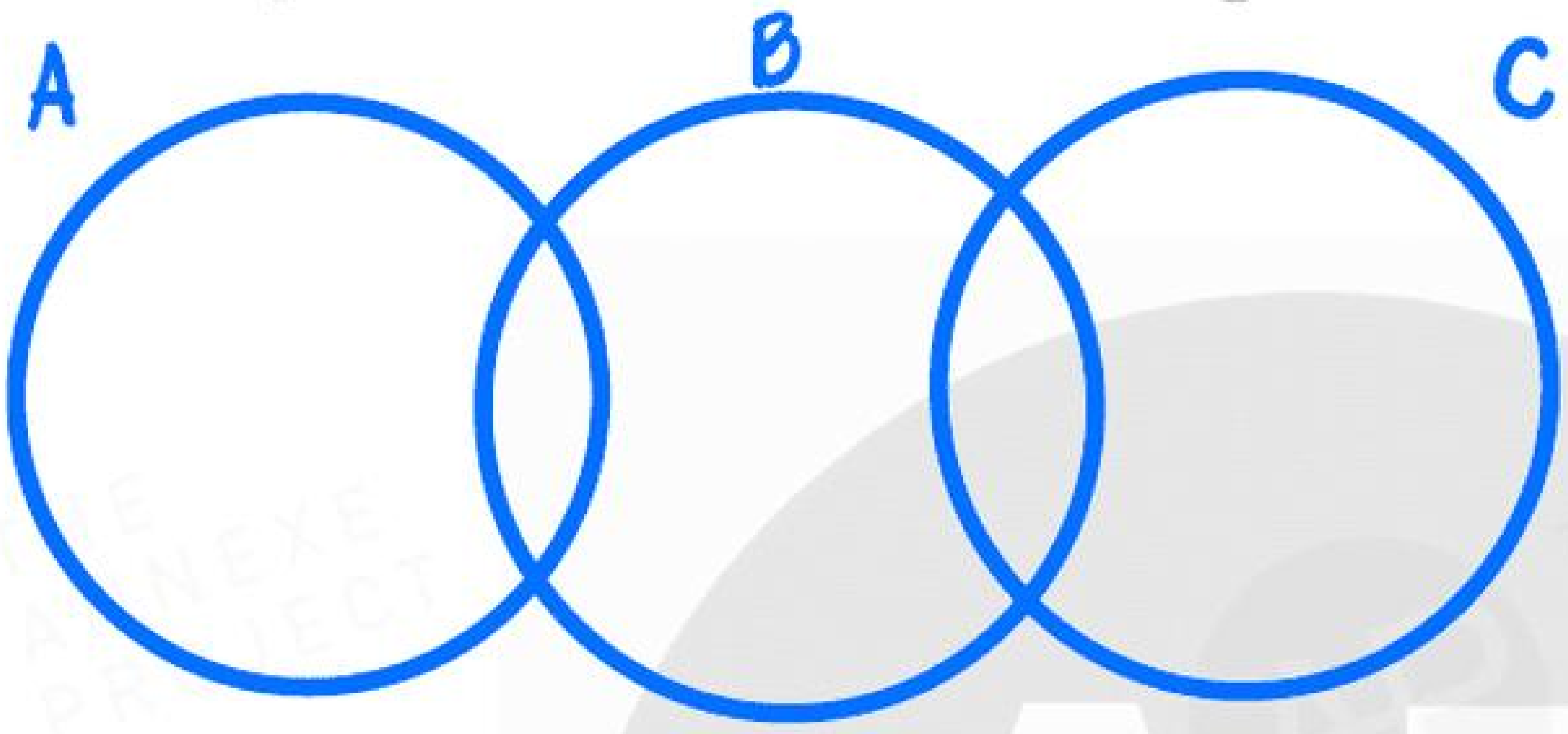
7 The events A, B and C are such that $\mathrm{P}(A) = a$, $\mathrm{P}(B) = b$ and $\mathrm{P}(C) = c$. A and B are independent events. A and C are mutually exclusive events.

(i) Find an expression for $\mathrm{P}(A' \cap B')$ and hence prove that A' and B' are independent events. [2]

(ii) Find an expression for $\mathrm{P}(A' \cap C')$. Draw a Venn diagram to illustrate the case when A' and C' are also mutually exclusive events. (You should not show event B on your diagram.) [2]

You are now given that A' and C' are **not** mutually exclusive, $\mathrm{P}(A) = \frac{2}{5}$, $\mathrm{P}(B \cap C) = \frac{1}{5}$ and $\mathrm{P}(A' \cap B' \cap C') = \frac{1}{10}$.

(iii) Find exactly the maximum and minimum possible values of $\mathrm{P}(A \cap B)$. [4]



(i) $P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - a - b + P(A \cap B)$

$= 1 - a - b + P(A) \cdot P(B)$

A & B are independent events, $\therefore P(A \cap B) = P(A).P(B)$

$= 1 - a - b + ab$

$= (1-a)(1-b)$

$= \underline{P(A') \cdot P(B')}$

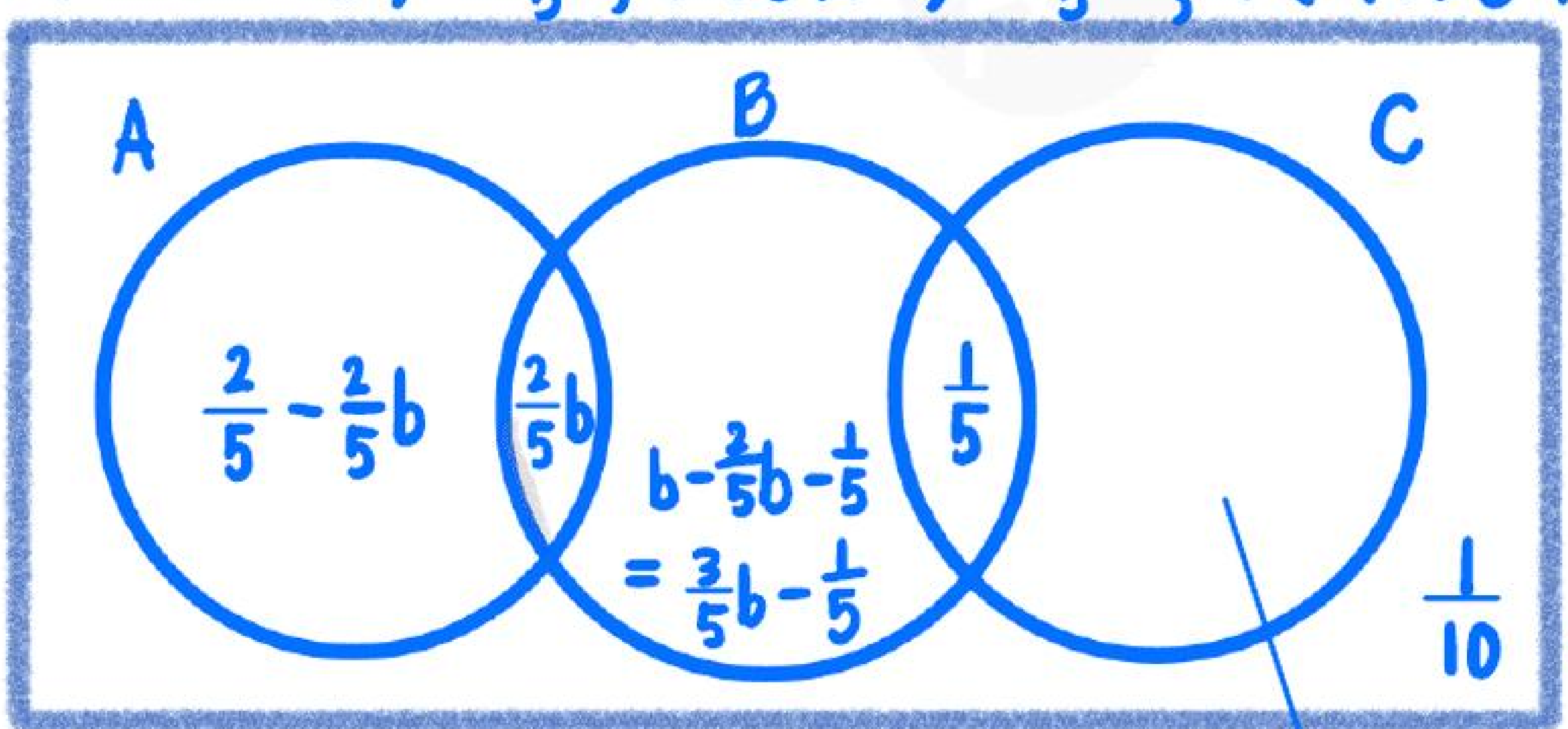
Since $P(A' \cap B') = P(A') \cdot P(B')$,
A' & B' are independent.

(ii) A and C are mutually exclusive, $P(A \cap C) = 0$

$$P(A' \cap C') = 1 - P(A \cup C)$$
$$= 1 - [P(A) + P(C)]$$
$$= \underline{1 - a - c}$$

If A' and C' are also mutually exclusive, then

(iii) Given $P(A) = \frac{2}{5}$, $P(B \cap C) = \frac{1}{5}$, $P(A' \cap B' \cap C') = \frac{1}{10}$



· Let $P(B) = b$

step 1:

· A and B are independent events,

$$\therefore P(A \cap B) = P(A) \cdot P(B)$$
$$= \frac{2}{5} P(B)$$
$$= \underline{\frac{2}{5} b}$$

step 2:

$$1 - \left[\frac{2}{5} - \frac{2}{5}b\right] - \frac{2}{5}b - \left[\frac{3}{5}b - \frac{1}{5}\right] - \frac{1}{5} - \frac{1}{10}$$
$$= \underline{\frac{1}{2} - \frac{3}{5}b}$$

step 3:

Consider: $\frac{3}{5}b - \frac{1}{5} \geq 0$

$$\underline{b \geq \frac{1}{3}}$$

Consider: $\frac{1}{2} - \frac{3}{5}b \geq 0$

$$\underline{b \leq \frac{5}{6}}$$

$\therefore \underline{\frac{1}{3} \leq b \leq \frac{5}{6}}$, then $\frac{1}{3}\left(\frac{2}{5}\right) \leq P(A \cap B) \leq \frac{5}{6}\left(\frac{2}{5}\right)$

$$\underline{\frac{2}{15} \leq P(A \cap B) \leq \frac{1}{3}}$$

8 A bag contains $(n+5)$ numbered balls. Two of the balls are numbered 3, three of the balls are numbered 4 and n of the balls are numbered 5. Two balls are taken, at random and without replacement, from the bag. The random variable S is the sum of the numbers on the two balls taken.

(i) Determine the probability distribution of S. [4]

(ii) For the case where $n = 1$, find $P(S = 10)$ and explain this result. [1]

(iii) Show that $E(S) = \dfrac{10n+36}{n+5}$ and $Var(S) = \dfrac{g(n)}{(n+5)^2(n+4)}$ where $g(n)$ is a quadratic polynomial to be determined. [6]

(i) $P(S=6) = \{3,3\}$

$$= \frac{2}{n+5} \times \frac{1}{n+4} = \frac{2}{(n+5)(n+4)}$$

$P(S=7) = \{3,4\}, \{4,3\}$

$$= \frac{2}{n+5} \times \frac{3}{n+4} \times 2 = \frac{12}{(n+5)(n+4)}$$

$P(S=8) = \{3,5\}, \{5,3\}, \{4,4\}$

$$= \frac{2}{n+5} \times \frac{n}{n+4} \times 2 + \frac{3}{n+5} \times \frac{2}{n+4}$$

$$= \frac{4n+6}{(n+5)(n+4)}$$

$P(S=9) = \{4,5\}, \{5,4\}$

$$= \frac{3}{n+5} \times \frac{n}{n+4} \times 2 = \frac{6n}{(n+5)(n+4)}$$

$P(S=10) = \{5,5\}$

$$= \frac{n}{n+5} \times \frac{n-1}{n+4} = \frac{n(n-1)}{(n+5)(n+4)}$$

r	6	7	8	9	10
$P(S=r)$	$\frac{2}{(n+5)(n+4)}$	$\frac{12}{(n+5)(n+4)}$	$\frac{4n+6}{(n+5)(n+4)}$	$\frac{6n}{(n+5)(n+4)}$	$\frac{n(n-1)}{(n+5)(n+4)}$

(ii) When $n=1$, $P(S=10) = 0$

It is not possible for S to be 10 as there is only 1 ball numbered '5'.

(iii) $$E(S) = 6 \times \frac{2}{(n+5)(n+4)} + 7 \times \frac{12}{(n+5)(n+4)} + 8 \times \frac{4n+6}{(n+5)(n+4)}$$

$$+ 9 \times \frac{6n}{(n+5)(n+4)} + 10 \times \frac{n(n-1)}{(n+5)(n+4)}$$

$$= \frac{12 + 84 + 32n + 48 + 54n + 10n(n-1)}{(n+5)(n+4)}$$

$$= \frac{144 + 76n + 10n^2}{(n+5)(n+4)}$$

$$= \frac{(10n+36)\cancel{(n+4)}}{(n+5)\cancel{(n+4)}}$$

$$= \frac{10n+36}{n+5}$$

$$Var(S) = E(S^2) - [E(S)]^2$$

$$= 6^2 \times \frac{2}{(n+5)(n+4)} + 7^2 \times \frac{12}{(n+5)(n+4)} + 8^2 \times \frac{4n+6}{(n+5)(n+4)}$$

$$+ 9^2 \times \frac{6n}{(n+5)(n+4)} + 10^2 \times \frac{n(n-1)}{(n+5)(n+4)} - \left[\frac{10n+36}{n+5}\right]^2$$

$$= \frac{72 + 588 + 256n + 384 + 486n + 100n^2 - 100n}{(n+5)(n+4)} - \frac{(10n+36)^2}{(n+5)^2}$$

$$= \frac{1044 + 642n + 100n^2}{(n+5)(n+4)} - \frac{100n^2 + 720n + 1296}{(n+5)^2}$$

$$= \frac{(n+5)(1044 + 642n + 100n^2) - (n+4)(100n^2 + 720n + 1296)}{(n+5)^2(n+4)}$$

$$= \frac{1044n + 642n^2 + \cancel{100n^3} + 5220 + 3210n + 500n^2 - \cancel{100n^3} - 720n^2 - 1296n - 400n^2 - 2880n - 5184}{(n+5)^2(n+4)}$$

$$= \frac{22n^2 + 78n + 36}{(n+5)^2(n+4)}$$

Hence, $g(n) = 22n^2 + 78n + 36$

$= 2(11n^2 + 39n + 18)$

$= \underline{2(11n+6)(n+3)}$

9 Many electronic devices need a fan to keep them cool. In order to maximise the lifetime of such fans, the speed they run at is reduced when conditions allow. Running a fan at a lower speed reduces the power required. The following table gives details, for a particular type of fan, of the power required (P watts) at different fan speeds (R revolutions per minute).

Fan speed (R)	3600	4500	5400	6300	7200	8100	9000	9900
Power (P)	0.22	0.34	0.52	0.78	1.06	1.48	2.04	2.64

(i) Draw a scatter diagram of these data. Use your diagram to explain whether the relationship between P and R is likely to be well modelled by an equation of the form $P = aR + b$, where a and b are constants. [2]

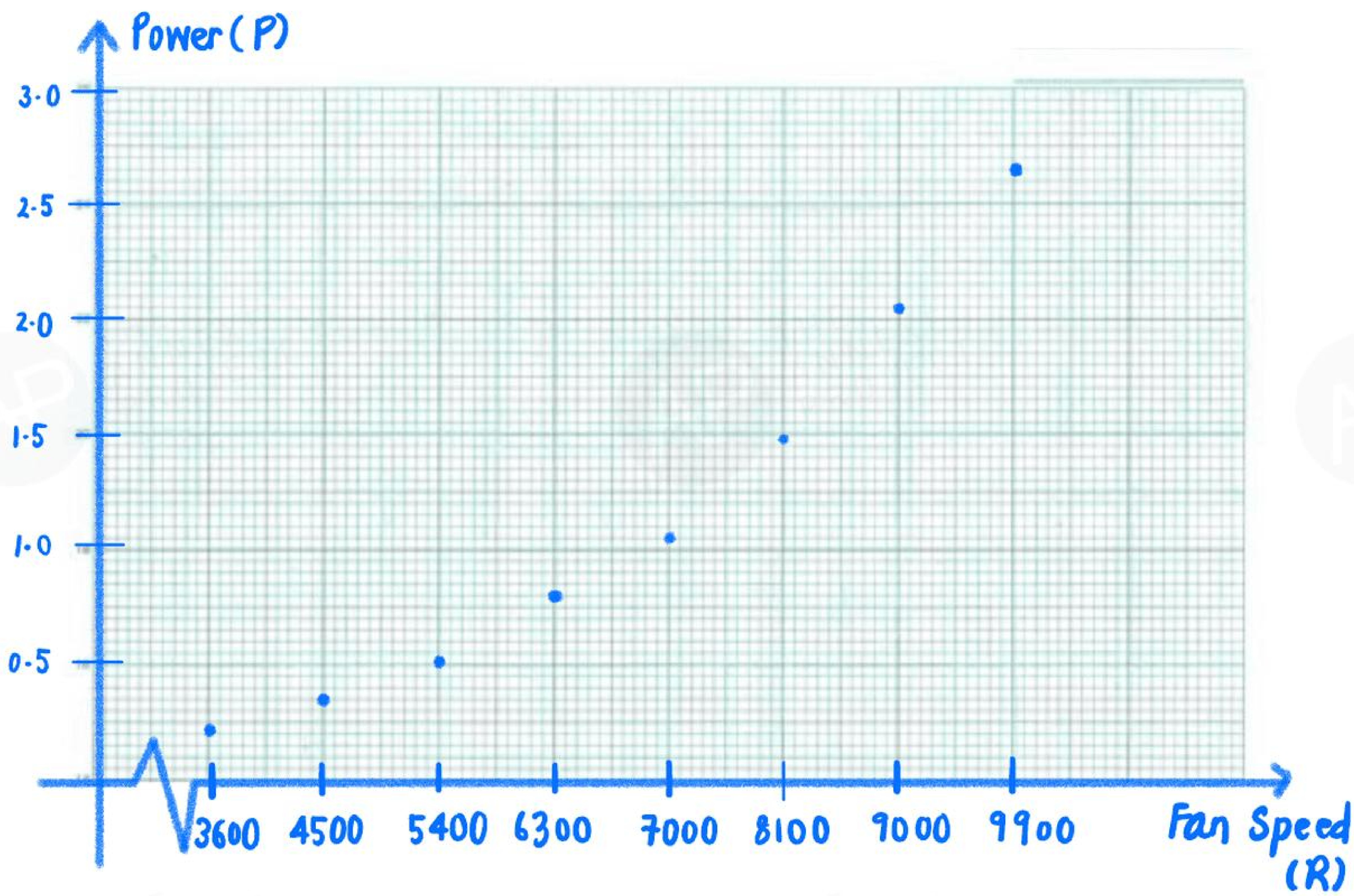
(ii) By calculating the relevant product moment correlation coefficients, determine whether the relationship between P and R is modelled better by $P = aR + b$ or by $P = aR^2 + b$. Explain how you decide which model is better, and state the equation in this case. [5]

(iii) Use your equation to estimate the speed of the fan when the power is 0.9 watts. Explain whether your estimate is reliable. [2]

(iv) Use your equation to estimate the power used when the speed of the fan is 3300 revolutions per minute. Explain whether your estimate is reliable. [2]

(v) Re-write your equation from part (ii) so that it can be used when the speed of the fan, R, is given in revolutions per second. [1]

(i)



As R increases, P increases at an increasing rate, hence it is unlikely to be modelled by $P = aR + b$ which is a linear model.

(ii) For $P = aR + b$:

$r = 0.969$

For $P = aR^2 + b$:

$r = 0.993$

Since $|r|$ between P and R^2 is closer to 1,
$P = aR^2 + b$ is a better model.
By GC: $P = 2.8457 \times 10^{-8} R^2 - 0.28261$

i.e. $P = 2.85 \times 10^{-8} R^2 - 0.283$

(iii) When $P = 0.9$:

$0.9 = 2.8457 \times 10^{-8} R^2 - 0.28261$

$R^2 = 41557788.94$

$R = 6446.5$

$= 6450$ rev/min.

The estimate is reliable because $P = 0.9$ is within the data range of P, and also the $|r|$ value between P and $R^2 \approx 1$.

(iv) When $R = 3300$:

$P = 2.8457 \times 10^{-8} (3300)^2 - 0.28261$

$= 0.027287$

$= 0.0273$ watts.

The estimate is unreliable because $R = 3300$ is outside the data range of R (extrapolation).

(v) 1 min = 60 seconds,
From part ii, $P = 2.8457 \times 10^{-8} R^2 - 0.28261$

Replace R with $60r$:

$\therefore P = 2.8457 \times 10^{-8} (60r)^2 - 0.28261$

$P = 1.0245 \times 10^{-4} r^2 - 0.28261$

$P = 1.02 \times 10^{-4} r^2 - 0.283$

10 In this question you should state the parameters of any distributions that you use.

A manufacturer produces specialist light bulbs. The masses in grams of one type of light bulb have the normal distribution $N(50, 1.5^2)$.

(i) Sketch the distribution for masses between 40 grams and 60 grams. [2]

(ii) Find the probability that the mass of a randomly chosen bulb is less than 50.4 grams. [1]

Each light bulb is packed into a randomly chosen box. The masses of the empty boxes have the distribution $N(75, 2^2)$.

(iii) Find the probability that the total mass of 4 randomly chosen empty boxes is more than 297 grams. [2]

(iv) Find the probability that the total mass of a randomly chosen light bulb and a randomly chosen box is between 124.9 and 125.7 grams. [3]

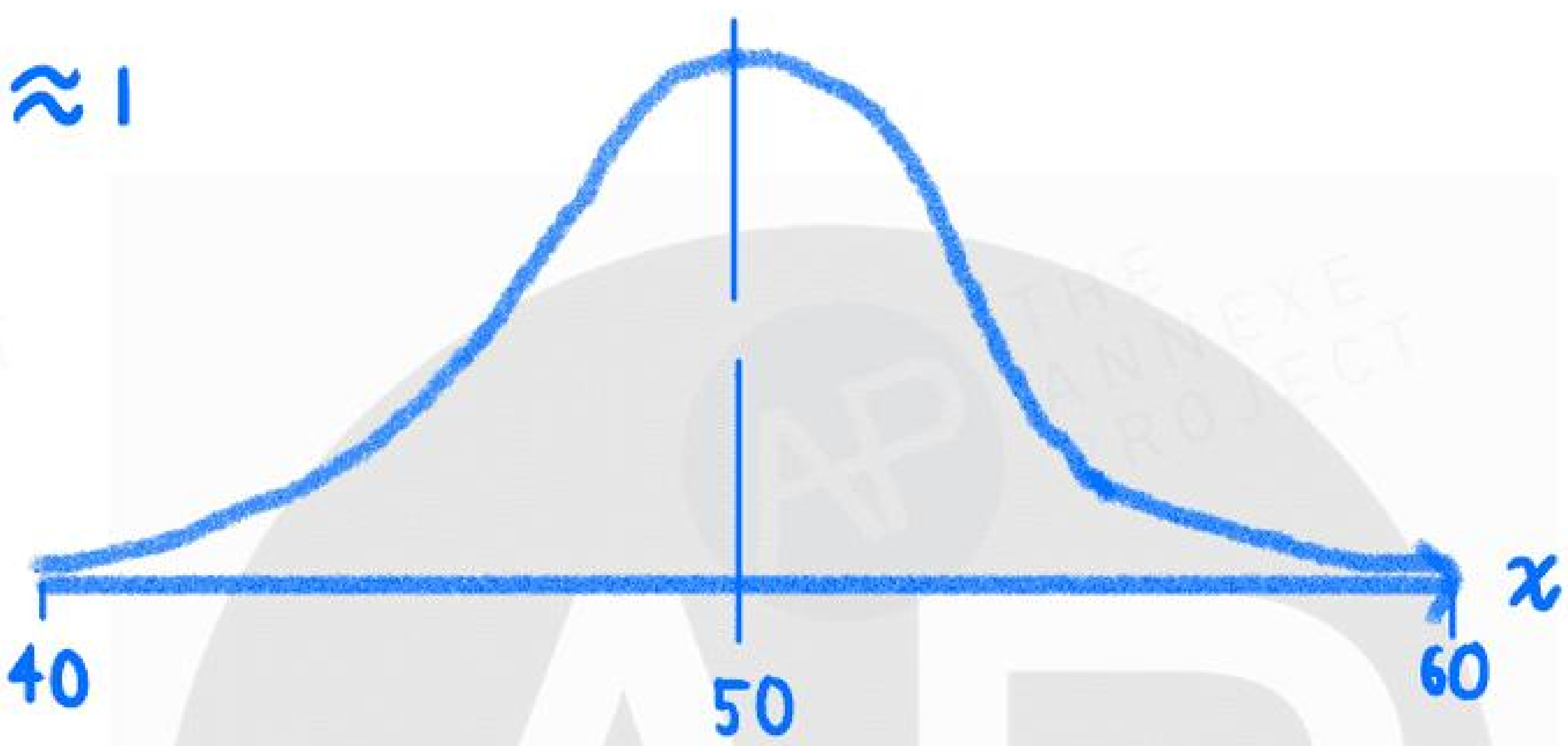
In order to protect the bulbs in transit each bulb is surrounded by padding before being packed in a box. The mass of the padding is modelled as 30% of the mass of the bulb.

(v) The probability that the total mass of a box containing a bulb and padding is more than k grams is 0.9. Find k. [4]

(vi) Find the probability that the total mass of 4 randomly chosen boxes, each containing a bulb and padding, is more than 565 grams. [3]

Let X be the mass (in g) of one type of light bulb. $X \sim N(50, 1.5^2)$

(i) By GC:
$P(40 < X < 60) \approx 1$



(ii) $P(X < 50.4) = 0.60514$
$= \underline{0.605}$

(iii) Let Y be the mass (in g) of an empty box. $Y \sim N(75, 2^2)$
$Y_1 + Y_2 + Y_3 + Y_4 \sim N(300, 16)$
$P(Y_1 + Y_2 + Y_3 + Y_4 > 297) = 0.77337$
$= \underline{0.773}$

(iv) $X+Y \sim N(125, 6.25)$

$P(124.9 < X+Y < 125.7) = 0.12621$
$= \underline{0.126}$

(v) Let T be the total mass of a box containing a bulb and padding.

$T = X + 0.3X + Y$

$E(T) = 50 + 15 + 75 = 140\ g$

$Var(T) = 1.5^2 + 0.3^2 \times 1.5^2 + 2^2 = 6.4525\ g^2$

$\therefore T \sim N(140, 6.4525)$

$P(T > K) = 0.9$

By GC, $K = 136.74$
$= \underline{137}$

(vi) $T_1 + T_2 + T_3 + T_4 \sim N(140 \times 4, 6.4525 \times 4)$

$P(T_1 + T_2 + T_3 + T_4 > 565) = 0.16251$
$= \underline{0.163}$