MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS **9758/02**

Paper 2 October/November 2019

3 hours

Section A: Pure Mathematics [40 marks]

1 You are given that $I = \int x(1-x)^{\frac{1}{2}}\,dx$.

(i) Use integration by parts to find an expression for I. [2]

(ii) Use the substitution $u^2 = 1 - x$ to find another expression for I. [2]

(iii) Show algebraically that your answers to parts (i) and (ii) differ by a constant. [2]

(i). Let $u = x$; Let $dv = (1-x)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ $\quad v = \frac{(1-x)^{\frac{3}{2}}}{(\frac{3}{2})(-1)} = -\frac{2}{3}(1-x)^{\frac{3}{2}}$

$$I = -\frac{2}{3}x(1-x)^{\frac{3}{2}} - \int -\frac{2}{3}(1-x)^{\frac{3}{2}}\,dx$$

$$= -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\left[\frac{(1-x)^{\frac{5}{2}}}{(\frac{5}{2})(-1)}\right] + C$$

$$= \underline{-\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + C}$$

(ii). If $u^2 = 1 - x$

$u = (1-x)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{1-x}} = \frac{-1}{2u}$

$$\text{then } I_1 = \int (1-u^2)u \cdot (-2u)\,du$$

$$= \int -2u^2(1-u^2)\,du$$

$$= \int -2u^2 + 2u^4\,du$$

$$= \frac{-2u^3}{3} + \frac{2u^5}{5} + c$$

$$= \underline{-\frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} + C_1}$$

(iii). $I_1 - I = \left[-\frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} + C_1\right] - \left[-\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + C\right]$

$$= (1-x)^{\frac{3}{2}}\left(\frac{2}{3}x - \frac{2}{3}\right) + \frac{2}{3}(1-x)^{\frac{5}{2}} + (C_1 - C)$$

$$= -\frac{2}{3}(1-x)^{\frac{5}{2}} + \frac{2}{3}(1-x)^{\frac{5}{2}} + (C_1 - C)$$

9 A company produces resistors rated at 750 ohms for use in electronic circuits. The production manager wishes to test whether the mean resistance of these resistors is in fact 750 ohms. He knows that the resistances are normally distributed with variance 100 ohms^2.

(i) Explain whether the manager should carry out a 1-tail test or a 2-tail test. State hypothesis for the test, defining any symbols you use. [2]

The production manager takes a random sample of 8 of these resistors. He finds that the resistances, in ohms, are as follows.

742 771 768 738 769 752 742 766

(ii) Find the mean of the sample of 8 resistors. Carry out the test, at the 5% level of significance, for the production manager. Give your conclusion in context. [5]

The company also produces resistors rated at 1250 ohms. Nothing is known about the distribution of the resistances of these resistors.

(iii) Describe how, and why, a test of the mean resistance of the 1250 ohms resistors would need to differ from that for the 750 ohms resistors. [2]

(i). He should conduct a 2-tail test since the resistance of these resistors can be less or more than 750 ohms.

$H_0 : \mu = 750\,\Omega$ against

$H_1 : \mu \neq 750\,\Omega$ where H_0 represents the null hypothesis, H_1 represents the alternative hypothesis, μ represents mean resistance.

(ii). $\bar{x} = 756$

Under H_0, using the Z-Test: $Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$= \frac{756-750}{\sqrt{\frac{100}{8}}}$$

$$= 1.6971$$

By GC, p-value = 0.089686

Since $p > 0.05$, there is insufficient evidence at 5% level of significance to reject H_0, i.e. the mean resistance of these resistors remain at $750\,\Omega$.

(iii). Since nothing is known about the distribution of the resistances of these resistors, a sample size of at least 30 such resistors must be randomly selected in order to approximate the mean distribution to be normally distributed by Central limit theorem.

10 Abi and Bhani find the fuel consumption for a car driven at different constant speeds. The table shows the fuel consumption, y kilometres per litre, for different constant speeds, x kilometres per hour.

x	40	45	50	55	60
y	22	20	18	17	16

(i) Abi decides to model the data using the line $y = 35 - \frac{1}{3}x$.

(a) On the grid opposite [2]

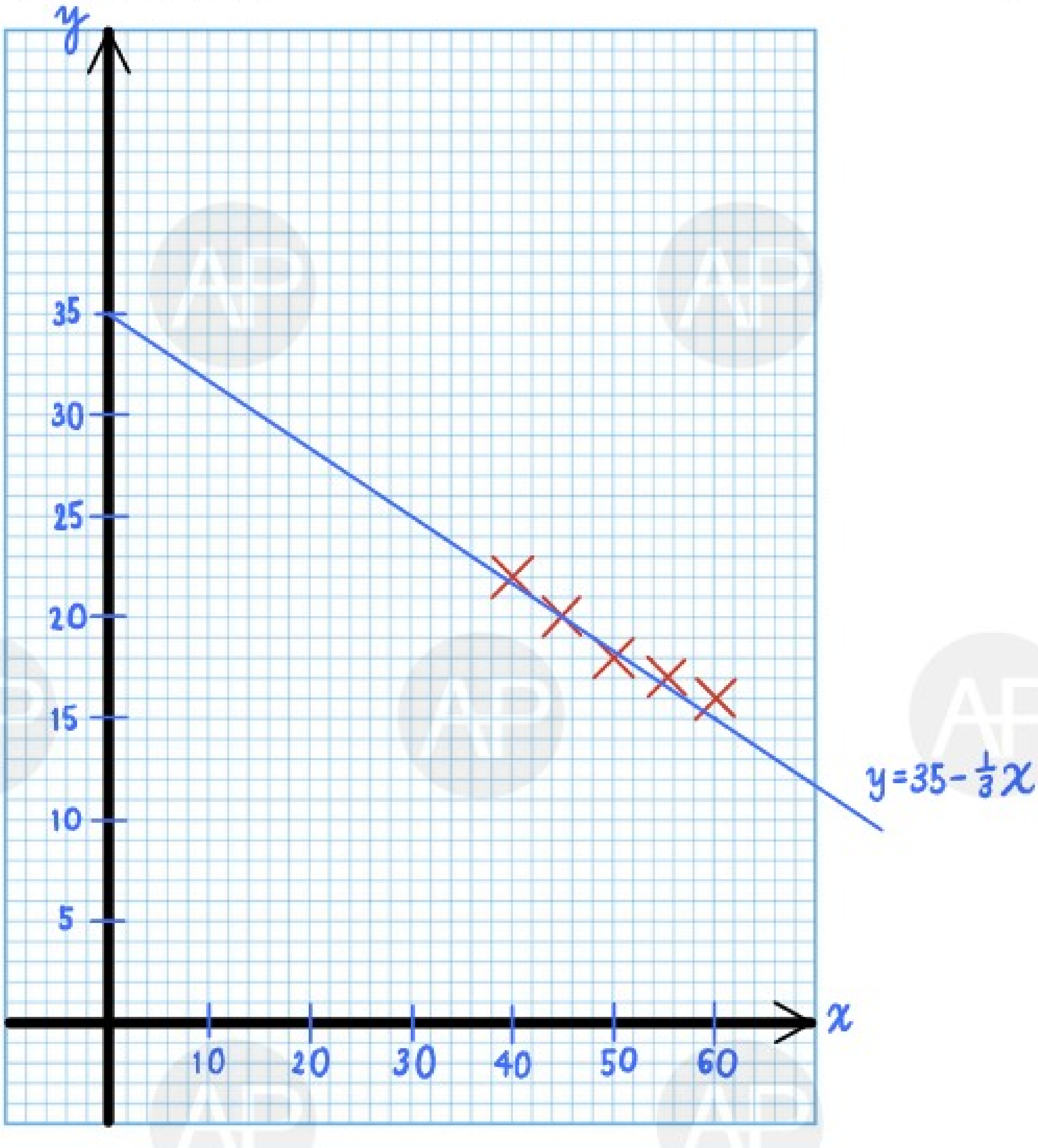
- draw a scatter diagram of the data,
- draw the line $y = 35 - \frac{1}{3}x$.

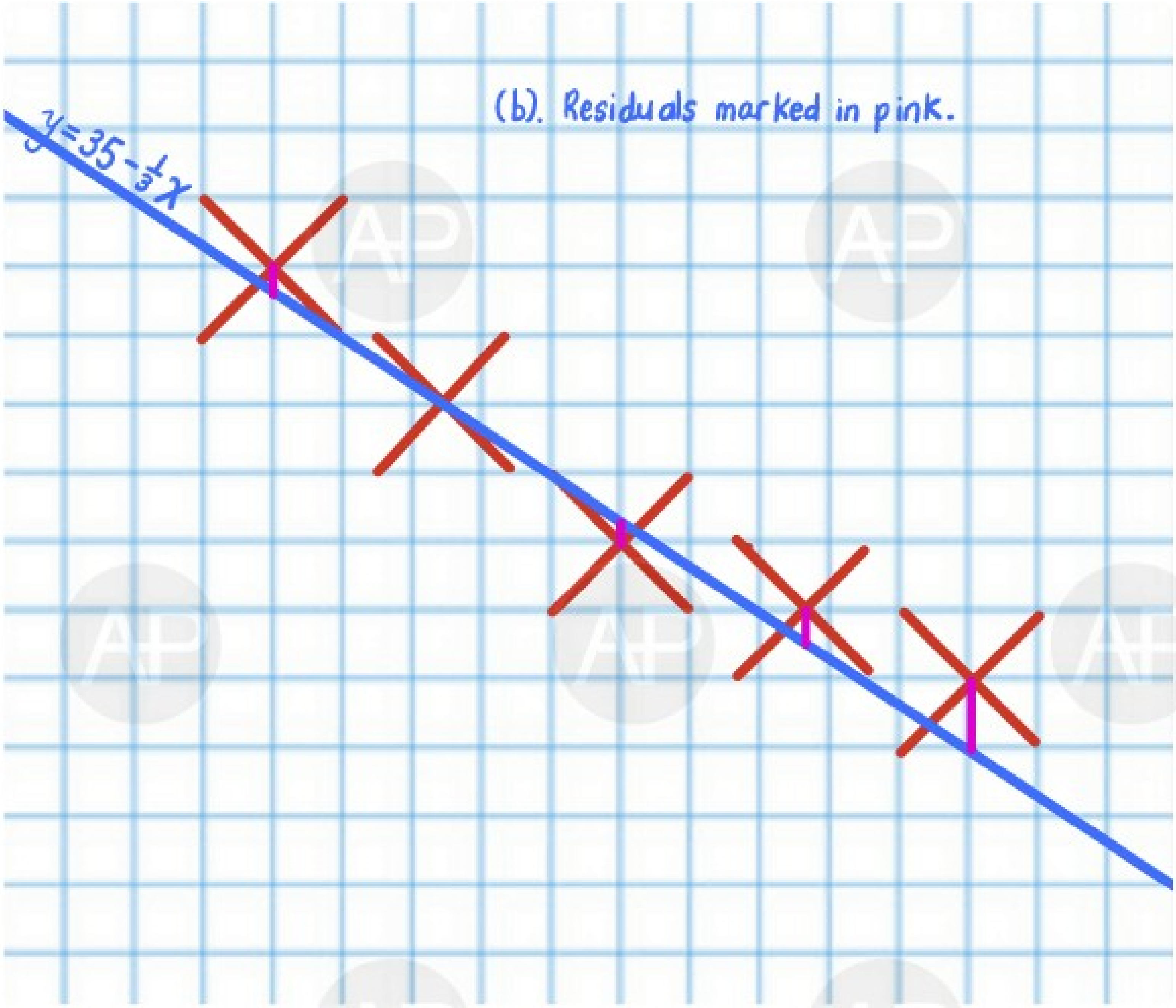
(b) For a line of best fit $y = \mathrm{f}(x)$, the residual for a point (a, b) plotted on the scatter diagram is the vertical distance between $(a, \mathrm{f}(a))$ and (a, b). Mark the residual for each point on your diagram. [1]

(c) Calculate the sum of the squares of the residuals for Abi's line. [1]

(d) Explain why, in general, the sum of the squares of the residuals rather than the sum of the residuals is used. [1]

(a).





(c). Sum of squares of residuals $= \left(\frac{1}{3}\right)^2 + 0^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 1^2$

$= \underline{\frac{4}{3}}$

(d). Because sum of residuals can be a negative value.

Bhani models the same data using a straight line passing through the points (40, 22) and (55, 17). The sum of the squares of the residuals for Bhani's line is 1.

(ii) State, with a reason, which of the two models, Abi's or Bhani's, gives a better fit. [1]

(iii) State the coordinates of the point that the least squares regression line must pass through. [1]

(iv) Use your calculator to find the equation of the least squares regression line of y on x. State the value of the product moment correlation coefficient. [3]

(v) Use the equation of the regression line to estimate the fuel consumption when the speed is 30 kilometres per hour. Explain whether you would expect this value to be reliable. [2]

(vi) Cerie performs a similar experiment on a different car. She finds that the sum of the squares of the residuals for her line is 0. What can you deduce about the data points in Cerie's experiment? [1]

(ii). Bhani's model. Because the sum of the squares of the residuals for Bhani's line is less than Abi's.

(iii). By GC, $\bar{x} = 50$ and $\bar{y} = 18.6$
$\therefore$ (50, 18.6) must lie on the least squares regression line.

(iv). By GC, $\underline{y = 33.6 - 0.3x}$, $r = -0.98480 = \underline{-0.985}$

(v). When $x = 30$ km/h , $y = \underline{24.6 \text{ km}/\ell}$
It is unreliable because $x = 30$km/h is outside of the range of data collected. Hence, extrapolation is unreliable.

(vi). All the data points lie on her line.

11 **In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

Arif is making models of hydrocarbon molecules. Hydrocarbons are chemical compounds made from carbon atoms and hydrogen atoms. Arif has a bag containing a large number of white balls to represent the carbon atoms, and a bag containing a large number of black balls to represent the hydrogen atoms. The masses of the white balls have the distribution $N(110, 4^2)$ and the masses of the black balls have the distribution $N(55, 2^2)$. The units for masses are grams.

(i) Find the probability that the total mass of 4 randomly chosen white balls is more than 425 grams. [2]

(ii) Find the probability that the total mass of a randomly chosen white ball and a randomly chosen black ball is between 161 and 175 grams. [2]

(iii) The probability that 2 randomly chosen white balls and 3 randomly chosen black balls have total mass less than M grams is 0.271. Find the value of M. [4]

(i). Let W be the r.v. denoting the mass of a white ball. $W \sim N(110, 4^2)$
Let B be the r.v. denoting the mass of a black ball. $B \sim N(55, 2^2)$

$W_1 + W_2 + W_3 + W_4 \sim N(440, 64)$

$P(W_1 + W_2 + W_3 + W_4 > 425) = 0.96960 = \underline{0.970}$

(ii). $W + B \sim N(165, 20)$

$P(161 < W + B < 175) = 0.80178 = \underline{0.802}$

(iii). $W_1 + W_2 + B_1 + B_2 + B_3 \sim N(385, 44)$

$P(W_1 + W_2 + B_1 + B_2 + B_3 < M) = 0.271$

$M = 380.96 = \underline{381\text{ g}}$

Arif also has a bag containing a large number of connecting rods to fix the balls together. The masses of the connecting rods, in grams, have the distribution N $(20, 0.9^2)$. In order to make models of methane (a hydrocarbon), Arif has to drill 1 hole in each black ball, and 4 holes in each white ball, for the connecting rods to fit in. This reduces the mass of each black ball by 10% and reduces the mass of each white ball by 30%.

A methane molecule consists of 1 carbon atom and 4 hydrogen atoms. Arif makes a model of a methane molecule using 4 black balls, 1 white ball and 4 connecting rods (see diagram above). The balls and connecting rods are all chosen at random.

(iv) Find the probability that the mass of Arif's model is more than 350 grams. [4]

Let R be the r.v. denoting the mass (in grams) of a connecting rod: $R \sim N(20, 0.9^2)$

$R_1 + R_2 + R_3 + R_4 + 0.9B_1 + 0.9B_2 + 0.9B_3 + 0.9B_4 + 0.7W \sim N(355, 24.04)$

P(mass of Arif's model > 350) $= 0.84608$

$= \underline{0.846}$

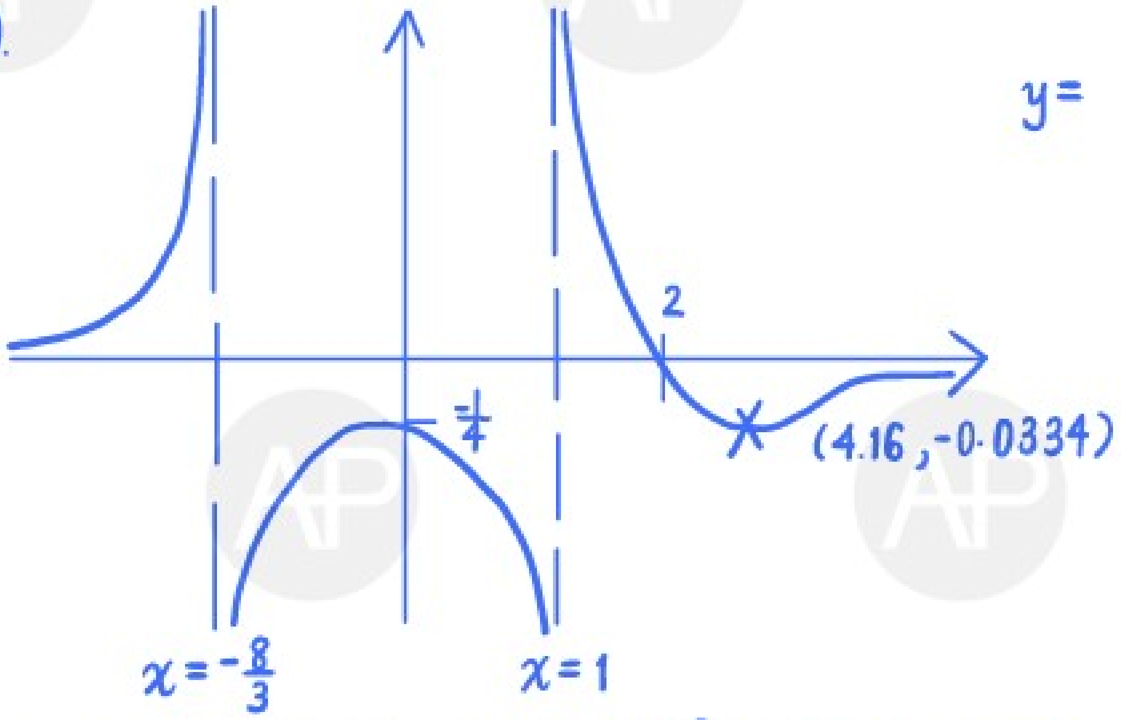
$= C_1 - C$ (Constant)

2 (i) Sketch the graph of $\frac{2-x}{3x^2+5x-8}$. Give the equations of the asymptotes and the coordinates of the point(s) where the curve crosses either axis. [4]

(ii) Solve the inequality $\frac{2-x}{3x^2+5x-8} > 0$. [1]

(iii) Hence solve the inequality $\frac{x-2}{3x^2+5x-8} > 0$. [1]

(i).



$$y = \frac{2-x}{(3x+8)(x-1)}$$

(ii). From graph above, $x < -\frac{8}{3}$ or $1 < x < 2$

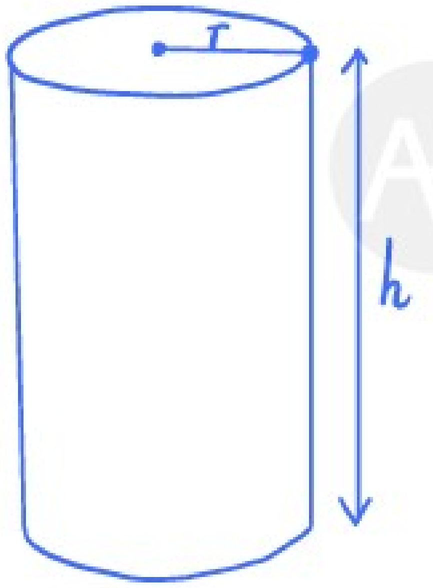
(iii). $\frac{x-2}{3x^2+5x-8} > 0$

$\therefore \frac{2-x}{3x^2+5x-8} < 0$

Hence, $-\frac{8}{3} < x < 1$ or $x > 2$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

3 A solid cylinder has radius r cm, height h cm and total surface area 900 cm^2. Find the exact value of the maximum possible volume of the cylinder. Find also the ratio $r : h$ that gives this maximum volume. [7]



$$A = 2\pi r^2 + 2\pi r h$$
$$900 = 2\pi r^2 + 2\pi r h$$
$$h = \frac{900 - 2\pi r^2}{2\pi r}$$

$$V = \pi r^2 h$$
$$= \cancel{\pi} r^{\cancel{2}} \left[\frac{900 - 2\pi r^2}{2\cancel{\pi}\cancel{r}} \right]$$
$$= 450r - \pi r^3$$
$$\frac{dV}{dr} = 450 - 3\pi r^2$$
Let $\frac{dV}{dr} = 0$
$$\therefore \quad 3\pi r^2 = 450$$
$$r = \sqrt{\frac{150}{\pi}} \text{ cm}$$

r	$\sqrt{\frac{150}{\pi}}^-$	$\sqrt{\frac{150}{\pi}}$	$\sqrt{\frac{150}{\pi}}^+$
$\frac{dV}{dr}$	/	—	\

When $r = \sqrt{\frac{150}{\pi}}$ cm, V is a max.

$$V_{max} = 450\sqrt{\frac{150}{\pi}} - \pi\left(\sqrt{\frac{150}{\pi}}\right)^3$$
$$= 450\sqrt{\frac{150}{\pi}} - \cancel{\pi}\left(\frac{150}{\cancel{\pi}}\right)\sqrt{\frac{150}{\pi}}$$
$$= 300\sqrt{\frac{150}{\pi}} = \underline{1500\sqrt{\frac{6}{\pi}} \text{ cm}^3}$$

When $r = \sqrt{\frac{150}{\pi}}$, $h = \frac{900 - 2\cancel{\pi}\left(\frac{150}{\cancel{\pi}}\right)}{2\pi\sqrt{\frac{150}{\pi}}} = \frac{600\sqrt{\pi}}{10\pi\sqrt{6}} = \frac{60}{\sqrt{6\pi}}$

$$\frac{r}{h} = \sqrt{\frac{150}{\pi}} \times \frac{\sqrt{6\pi}}{60} = \frac{5\sqrt{6}}{\cancel{\sqrt{\pi}}} \times \frac{\sqrt{6}\cancel{\sqrt{\pi}}}{60} = \underline{\frac{1}{2}}$$

4 (i) Given that $\mathrm{f}(x) = \sec 2x$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$. Hence, or otherwise, find the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^2. [5]

(ii) Use your series from part (i) to estimate $\int_0^{0.02} \sec 2x \, dx$, correct to 5 decimal places. [2]

(iii) Use your calculator to find $\int_0^{0.02} \sec 2x \, dx$, correct to 5 decimal places. [1]

(iv) Comparing your answers to parts (ii) and (iii), and with reference to the value of x, comment on the accuracy of your approximations. [2]

(v) Explain why a Maclaurin series for $\mathrm{g}(x) = \operatorname{cosec} 2x$ cannot be found. [1]

(i). $f(x) = \sec 2x$

$f'(x) = 2\sec 2x \tan 2x$

$f''(x) = 2\sec 2x \cdot \sec^2 2x\,(2) + \tan 2x \cdot 2\sec 2x \tan 2x \cdot 2$

$= 4\sec 2x\,(\sec^2 2x + \tan^2 2x)$

$= 4\sec 2x\,(2\sec^2 2x - 1)$

When $x = 0$, $f(x) = 1$

$f'(x) = 0$

$f''(x) = 4$

Hence, $f(x) = 1 + \frac{x^2}{2!}(4) + \cdots$

$= 1 + 2x^2 + \cdots$

(ii). $\int_0^{0.02} \sec 2x \, dx \approx \int_0^{0.02} 1 + 2x^2 \, dx$

$= \left[x + \frac{2x^3}{3} \right]_0^{0.02}$

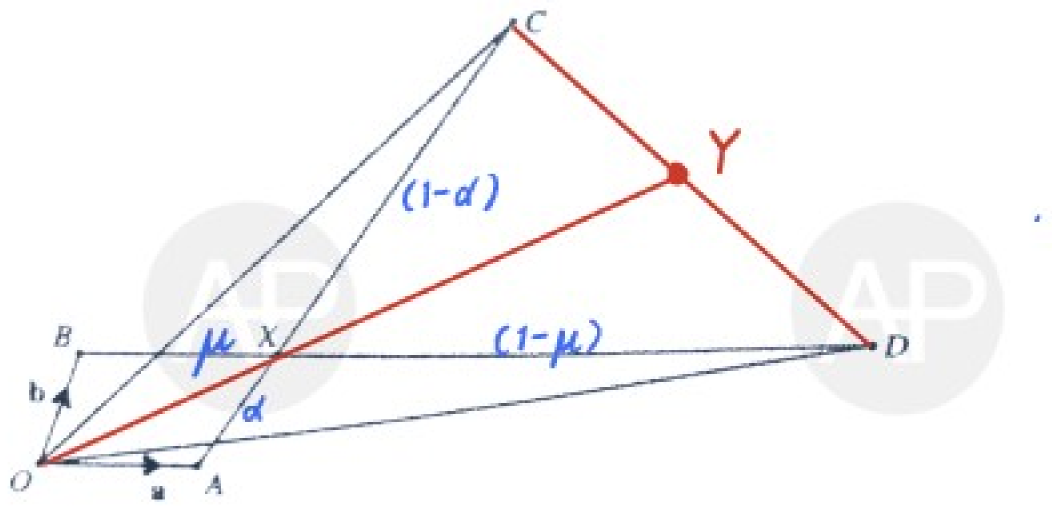
$= 0.02001$

(iii). By GC, $\int_0^{0.02} \sec 2x \, dx = 0.02001$

(iv). Since $x = 0.02$ is small and close to 0, the approximation in (ii) is accurate.

(v). $g(0)$ is undefined.

5



With reference to the origin O, the points A, B and C are such that $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OC} = 2\boldsymbol{a} + 4\boldsymbol{b}$ and $\overrightarrow{OD} = \boldsymbol{b} + 5\boldsymbol{a}$. The lines **BD** and **AC** cross at X (see diagram).

(i) Express $\overrightarrow{OX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [4]

The point Y lies on **CD** and is such that the points O, X and Y are collinear.

(ii) Express $\overrightarrow{OY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ and find the ratio $OX : OY$. [6]

(i). Let $BX : XD = \mu : 1-\mu$

Let $AX : XC = \alpha : 1-\alpha$

By ratio theorem,

$$\overrightarrow{OX} = \mu\overrightarrow{OD} + (1-\mu)\overrightarrow{OB}$$
$$= \mu(b+5a) + (1-\mu)b$$
$$= 5\mu a + b$$
$$\overrightarrow{OX} = \alpha\overrightarrow{OC} + (1-\alpha)\overrightarrow{OA}$$
$$= \alpha(2a+4b) + (1-\alpha)a$$
$$= (1+\alpha)a + 4\alpha b$$

By comparison of coefficients:

$5\mu = 1+\alpha$ ——— (1)

$4\alpha = 1$

$\therefore \alpha = \frac{1}{4}$ ——— (2)

$\Rightarrow \mu = \frac{1}{4}$

Hence, $\overrightarrow{OX} = \frac{5}{4}a + b$

(ii). let $\overrightarrow{OY} = \lambda\overrightarrow{OX}$

$$= \lambda\left(\frac{5}{4}a + b\right)$$
$$= \frac{5}{4}\lambda a + \lambda b$$

also $\overrightarrow{CY} = s\,\overrightarrow{CD}$

$$\overrightarrow{OY} - (2a+4b) = s(b+5a-2a-4b)$$
$$\overrightarrow{OY} = s(3a-3b) + 2a + 4b$$
$$= (3s+2)a + (4-3s)b$$

By Comparison of coefficients:

$3s+2 = \frac{5}{4}\lambda$

$12s+8 = 5\lambda$

$\lambda = \frac{12s+8}{5}$ ——— (3)

$4-3s = \lambda$ ——— (4)

$\frac{12s+8}{5} = 4-3s$

$12s+8 = 20-15s$

$27s = 12$

$s = \frac{4}{9}$, $\lambda = \frac{8}{3}$

$\therefore \overrightarrow{OY} = \frac{5}{4}\left(\frac{8}{3}\right)a + \frac{8}{3}b$

$= \frac{10}{3}a + \frac{8}{3}b$

$\overrightarrow{OX} : \overrightarrow{OY} = \left(\frac{5}{4}a + b\right) : \left(\frac{10}{3}a + \frac{8}{3}b\right)$

$= \left(\frac{5}{4}a + b\right) : \frac{8}{3}\left(\frac{5}{4}a + b\right)$

$= 1 : \frac{8}{3}$

$= 3 : 8$

Section B: Probability and Statistics [60 marks]

6 In a certain country there are 100 professional football clubs, arranged in 4 divisions. There are 22 clubs in Division One, 24 in Division Two, 26 in Division Three and 28 in Division Four.

(i) Alice wishes to find out approaches to training by clubs in Division One, so she sends a questionnaire to the 22 clubs in Division One. Explain whether these 22 clubs form a sample or a population. [1]

(ii) Dilip wishes to investigate the facilities for supporters at the football clubs, but does not want to obtain the detailed information necessary from all 100 clubs. Explain how he should carry out his investigation, and why he should do the investigation this way. [2]

(iii) Find the number of different possible samples of 20 football clubs, with 5 clubs chosen from each division. [3]

(i). Sample. These 22 clubs are part of the 100 clubs (population).

(ii). Randomly select 2 clubs from Division 1, 2 clubs from Division 2, 3 clubs from Division 3 and 3 clubs from Division 4.
This way, he can investigate the facilities for supporters in all 4 divisions in an unbiased and representative (proportional) way.

(iii). $^{22}C_5 \times {}^{24}C_5 \times {}^{26}C_5 \times {}^{28}C_5 = \underline{7.24 \times 10^{18} \text{ ways}}$

7 A company produces drinking mugs. It is known that, on average, 8% of the mugs are faulty. Each day the quality manager collects 50 of the mugs at random and checks them; the number of faulty mugs found is the random variable F.

(i) State, in the context of the question, two assumptions needed to model F by a binomial distribution. [2]

You are now given that F can be modelled by a binomial distribution.

(ii) Find the probability that, on a randomly chosen day, at least 7 faulty mugs are found. [2]

(iii) The number of faulty mugs produced each day is independent of other days. Find the probability that, in a randomly chosen working week of 5 days, at least 7 faulty mugs are found on no more than 2 days. [2]

(i). The probability of each randomly chosen mug is faulty is constant.

There are only 2 mutually exclusive outcomes: Either the mugs are faulty or they are not faulty.

(ii). Let F be the discrete r.v. denoting the no. of faulty mugs out of 50:

$$F \sim B(50, 0.08)$$

$$P(F \geq 7) = 1 - P(F \leq 6) = 0.10187 = \underline{0.102}$$

(iii). Let W be the no. of days out of 5 that at least 7 faulty mugs are found on each day:

$$W \sim B(5, 0.10187)$$

$$P(W \leq 2) = 0.99098 = \underline{0.991}$$

The company also makes saucers. The number of faulty saucers also follows a binomial distribution. The probability that a saucer is faulty is p. Faults on saucers are independent of faults on mugs.

(iv) Write down an expression in terms of p for the probability that, in a random sample of 10 saucers, exactly 2 are faulty. [1]

The mugs and saucers are sold in sets of 2 randomly chosen mugs and 2 randomly chosen saucers. The probability that a set contains at most 1 faulty item is 0.97.

(v) Write down an equation satisfied by p. Hence find the value of p. [4]

(iv). Let X be the no. of faulty saucers out of 10 saucers:

$$X \sim B(10, p)$$

$$P(X=2) = \binom{10}{2} p^2 (1-p)^8$$

$$= 45p^2(1-p)^8$$

(v). P(1 saucer faulty only) + P(1 mug faulty only) + P(no faulty items) = 0.97

$$[{}^2C_1 \times p(1-p)](0.92)^2 + [{}^2C_1 \times 0.08 \times 0.92](1-p)^2 + (0.92)^2(1-p)^2 = 0.97$$

$$1.6928p(1-p) + 0.1472(1-p)^2 + 0.8464(1-p)^2 = 0.97$$

By GC: $p = 0.068891 = \underline{0.0689}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

8 Gerri collects characters given away in packets of breakfast cereal. There are four different characters: Horse, Rider, Dog and Bird. Each character is made in four different colours: Orange, Yellow, Green and White. Gerri has collected 56 items; the numbers of each character and colour are shown in the table.

	Orange	Yellow	Green	White
Horse	1	1	3	4
Rider	1	1	7	5
Dog	3	7	1	6
Bird	4 x	5 y	6	1

(i) Gerri puts all the items in a bag and chooses one item at random.

(a) Find the probability that this item is either a Horse or a Rider. [1]

(b) Find the probability that this item is either a Dog or a Bird but the item is not White. [1]

(ii) Gerri now puts the item back in the bag and chooses two items at random.

(a) Find the probability that both of the items are Horses, but neither of the items is Orange. [1]

(b) Find the probability that Gerri's two items include exactly one Dog and exactly one item that is Yellow. [3]

(iii) Gerri has two favourites among the 16 possible colour / character combinations. The probability of choosing these two at random from the 56 items is $\frac{1}{77}$. Write down all the possibilities for Gerri's two favourite colour / character combinations. [3]

(i). (a). P(either a Horse or a Rider) $= \frac{23}{56}$

(b). P(either a Dog or a Bird but not white) $= \frac{26}{56} = \frac{13}{28}$

(ii). (a). P(Both Horses but neither orange) $= \frac{8}{56} \times \frac{7}{55} = \frac{1}{55}$

(b). [1 Yellow Dog] + [1 Yellow item (not dog) + non-yellow dog]

$$= 2\left(\frac{7}{56} \times \frac{32}{55}\right) + 2\left(\frac{7}{56} \times \frac{10}{55}\right) = 2\left(\frac{4}{55} + \frac{1}{44}\right) = \frac{21}{110}$$

(iii). Let x and y be any of the 2 numbers in the above 16 possible combinations:

$$2\left(\frac{x}{56} \times \frac{y}{55}\right) = \frac{1}{77}$$

$$xy = 20$$

i.e. possible combinations include $\{$ orange bird + yellow bird,
white horse + white rider,