1 (i) Differentiate $x^2 \ln x$ with respect to x. [2]

$y = x^2 \ln x$

$\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + \ln x\,(2x)$

$= x(1 + 2\ln x)$

(ii) Hence find $\int x \ln x \, dx$. [3]

$\frac{1}{2}\int 2x \ln x \; dx$

$= \frac{1}{2}\int x + 2x\ln x - x \; dx$

$= \frac{1}{2}\left[x^2 \ln x - \frac{x^2}{2}\right] + c$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

2 (i) Prove that $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{1}{\sin\theta} = \cot\theta.$ [4]

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\sin^2\theta - 1 + \cos\theta}{(1-\cos\theta)(\sin\theta)}$$

$$= \frac{(\cancel{1} - \cos^2\theta) + \cos\theta - \cancel{1}}{(1-\cos\theta)(\sin\theta)}$$

$$= \frac{\cos\theta\,\cancel{(1-\cos\theta)}}{\cancel{(1-\cos\theta)}(\sin\theta)}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

$$= \text{RHS}$$

(ii) Hence solve the equation $\frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta} = 3\tan\theta$ for $0° \leqslant \theta \leqslant 180°$. [3]

$\therefore \cot\theta = 3\tan\theta$

$\frac{1}{\tan\theta} = 3\tan\theta$

$3\tan^2\theta = 1$

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

Basic Angle for $\theta = 30°$

since θ lies in the 1st or 2nd quadrant,

$\theta = 30°$ or $150°$

3 (i) Show that $x-1$ is a factor of x^3+x^2-x-1. [1]

Let $f(x) = x^3+x^2-x-1$

$f(1) = 1+1-1-1 = 0$

$\therefore (x-1)$ is a factor of x^3+x^2-x-1.

(ii) Express $\dfrac{4}{x^3+x^2-x-1}$ as the sum of three partial fractions. [7]

$$\begin{array}{r}
x^2+2x+1 \\
x-1 \,\big)\, \overline{x^3+x^2-x-1} \\
-(x^3-x^2) \\
\hline
2x^2-x-1 \\
-(2x^2-2x) \\
\hline
x-1 \\
-(x-1) \\
\hline
0
\end{array}$$

$x^3+x^2-x-1 = (x-1)(x^2+2x+1)$

$= (x-1)(x+1)^2$

$\therefore \dfrac{4}{x^3+x^2-x-1} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2}$

$4 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

Sub $x=1$: $4 = 4A$

$A = 1$ //

Sub $x=-1$: $4 = -2C$

$C = -2$ //

Sub $x=0$: $4 = 1 + B(-1) - 2(-1)$

$4 = 1 - B + 2$

$B = -1$ //

Hence, $\dfrac{4}{x^3+x^2-x-1} = \underline{\underline{\dfrac{1}{x-1} - \dfrac{1}{x+1} - \dfrac{2}{(x+1)^2}}}$

4 **(a)** Solve the equation $\log_2 x + \log_{16} x = -2.5$. [3]

$$\log_2 x + \log_{16} x = -2.5$$
$$\log_2 x + \frac{\log_2 x}{\log_2 16} = -2.5$$
$$\log_2 x + \frac{\log_2 x}{4} = -2.5$$
$$4\log_2 x + \log_2 x = -10$$
$$5\log_2 x = -10$$
$$\log_2 x = -2$$
$$x = 2^{-2}$$
$$= \frac{1}{4}$$

(b) It is given that $\lg z - \lg y = \lg(z+y)$.

(i) Express z in terms of y. [3]

$$\lg \frac{z}{y} = \lg(z+y)$$
$$\therefore \frac{z}{y} = z + y$$
$$z = zy + y^2$$
$$z - zy = y^2$$
$$z(1-y) = y^2$$
$$\therefore z = \frac{y^2}{1-y}$$

(ii) State the range of values of z and explain clearly why $0 < y < 1$. [2]

For $\lg z$ to exist, $z > 0$.

Hence, $\frac{y^2}{1-y} > 0$

Since $y^2 \geq 0$, $1 - y > 0$

i.e. $y < 1$ ——— (1)

also, for $\lg y$ to exist, $y > 0$ ——— (2)

Combining (1) & (2): $\underline{0 < y < 1}$

5 $f(x)$ is such that $f''(x) = 3\cos 3x - 4\sin 2x$. Given that $f(0) = 0$ and $f\left(\frac{\pi}{2}\right) = \frac{5}{6}$, show that $f\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2}$.

[9]

$$f''(x) = 3\cos 3x - 4\sin 2x$$

$$\therefore f'(x) = \int 3\cos 3x - 4\sin 2x \, dx$$

$$= \frac{3\sin 3x}{3} + \frac{4\cos 2x}{2} + C$$

$$= \sin 3x + 2\cos 2x + C$$

$$f(x) = \int \sin 3x + 2\cos 2x + C \, dx$$

$$= \frac{-\cos 3x}{3} + \frac{2\sin 2x}{2} + Cx + D$$

$$= -\frac{1}{3}\cos 3x + \sin 2x + Cx + D$$

Given $f(0) = 0$

then $-\frac{1}{3}\cos 0 + D = 0$

$D = \frac{1}{3}$ //

Given $f\left(\frac{\pi}{2}\right) = \frac{5}{6}$

then $-\frac{1}{3}\cos\frac{3\pi}{2} + \sin\pi + \frac{\pi}{2}C + \frac{1}{3} = \frac{5}{6}$

$\frac{\pi}{2}C = \frac{1}{2}$

$\therefore C = \frac{1}{\pi}$

Hence, $f(x) = \frac{-1}{3}\cos 3x + \sin 2x + \frac{1}{\pi}x + \frac{1}{3}$

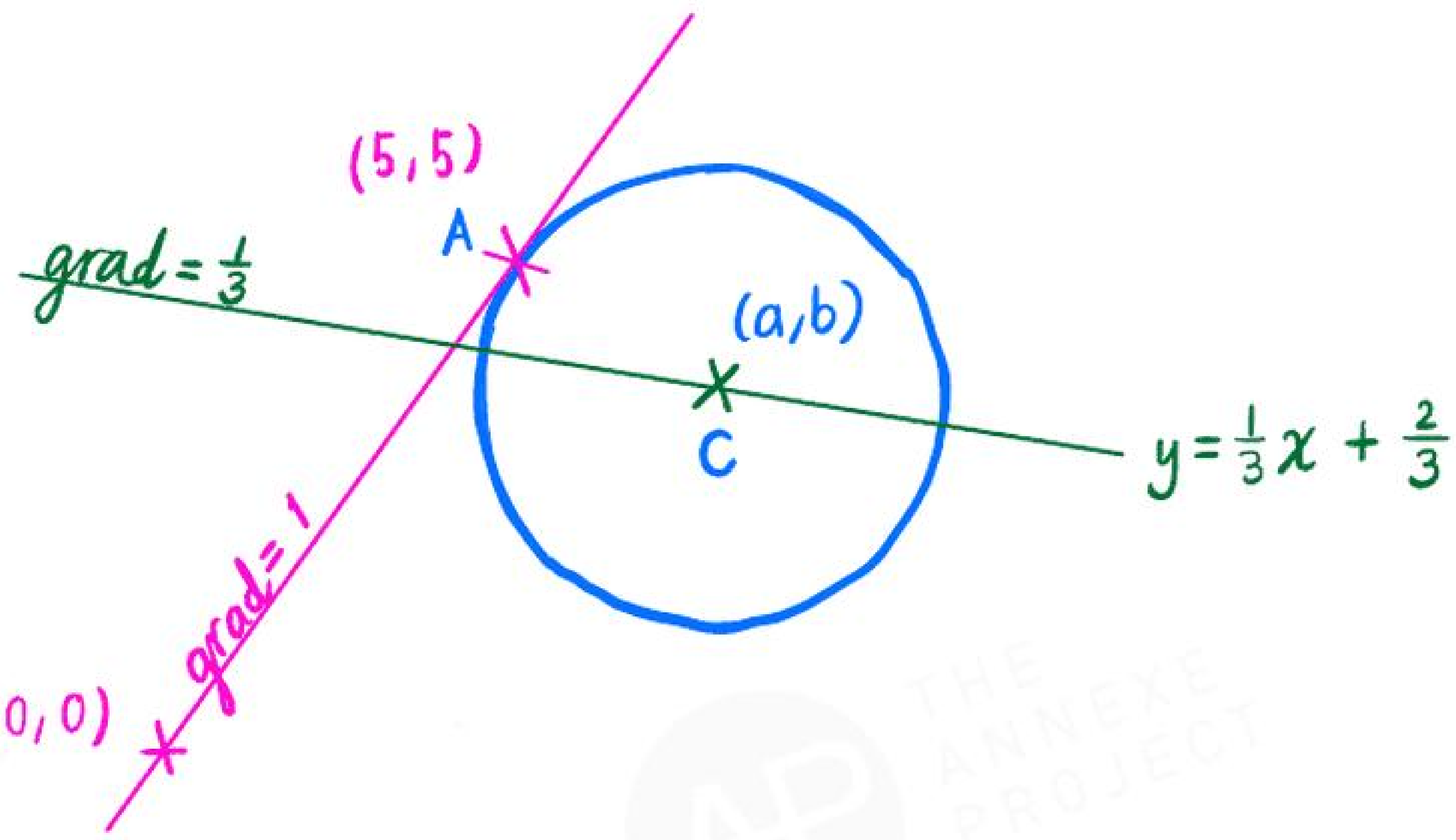
$\Longrightarrow f\left(\frac{\pi}{3}\right) = \frac{-1}{3}\cos\pi + \sin\frac{2\pi}{3} + \frac{1}{\pi}\left(\frac{\pi}{3}\right) + \frac{1}{3}$

$= \frac{1}{3} + \frac{\sqrt{3}}{2} + \frac{1}{3} + \frac{1}{3}$

$= 1 + \frac{\sqrt{3}}{2}$ (shown).

6 A tangent to a circle at the point (5, 5) passes through the origin. The line with equation $3y = x+2$ is a normal to the circle.

(i) Showing all your working, find the equation of the circle. [7]



Let the centre of circle be (a,b):

gradient $AC = -1$

$\frac{b-5}{a-5} = -1$

$b-5 = 5-a$

$b = 10 - a$ —— (1)

(a,b) lies on $y = \frac{1}{3}x + \frac{2}{3}$

$\therefore b = \frac{1}{3}a + \frac{2}{3}$

$3b = a + 2$ —— (2)

Sub (1) into (2):

$3(10-a) = a+2$

$30 - 3a = a + 2$

$4a = 28$

$a = 7$

$\therefore b = 3$

centre of circle $= (7, 3)$

radius of circle

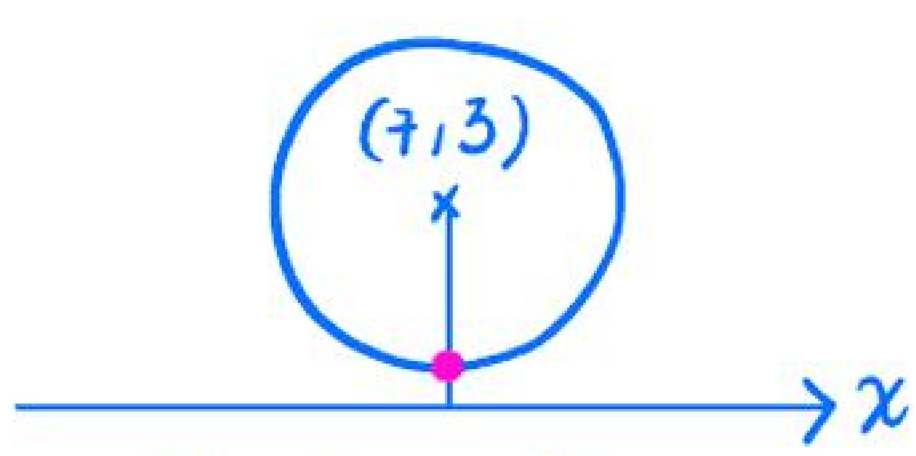
$= \sqrt{(3-5)^2 + (7-5)^2}$

$= \sqrt{4+4} = 2\sqrt{2}$ units.

$\therefore$ Equation of circle

: $(x-7)^2 + (y-3)^2 = 8$

(ii) Find the coordinates of the point on the circle which is nearest to the x-axis. [2]

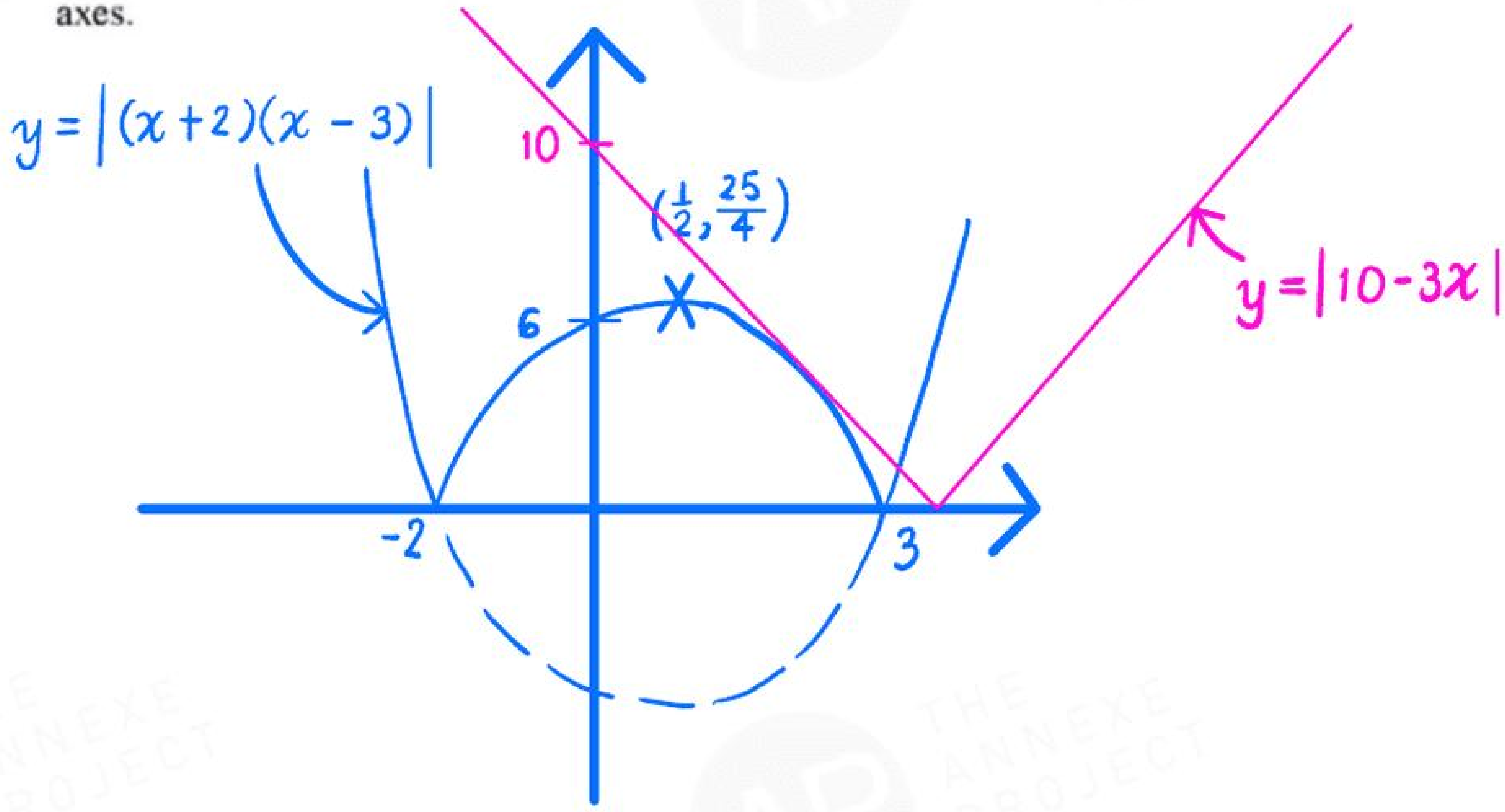


x - coordinate $= 7$

y - coordinate $= 3 - 2\sqrt{2}$

$\Rightarrow (7,\ 3 - 2\sqrt{2})$

7 (i) Sketch the graph of $y = |x^2 - x - 6|$, showing the coordinates of the points where the curve meets the axes. [5]



A line, with an equation of the form $y = k - 3x$, is a tangent to the curve $y = |x^2 - x - 6|$ at the point where $x = 2$.

(ii) Show that $k = 10$. [2]

When $x = 2$, $y = |2^2 - 2 - 6|$
$= 4$

Sub $(2, 4)$ into $y = k - 3x$,
$4 = k - 6$
$\therefore \underline{\underline{k = 10}}$

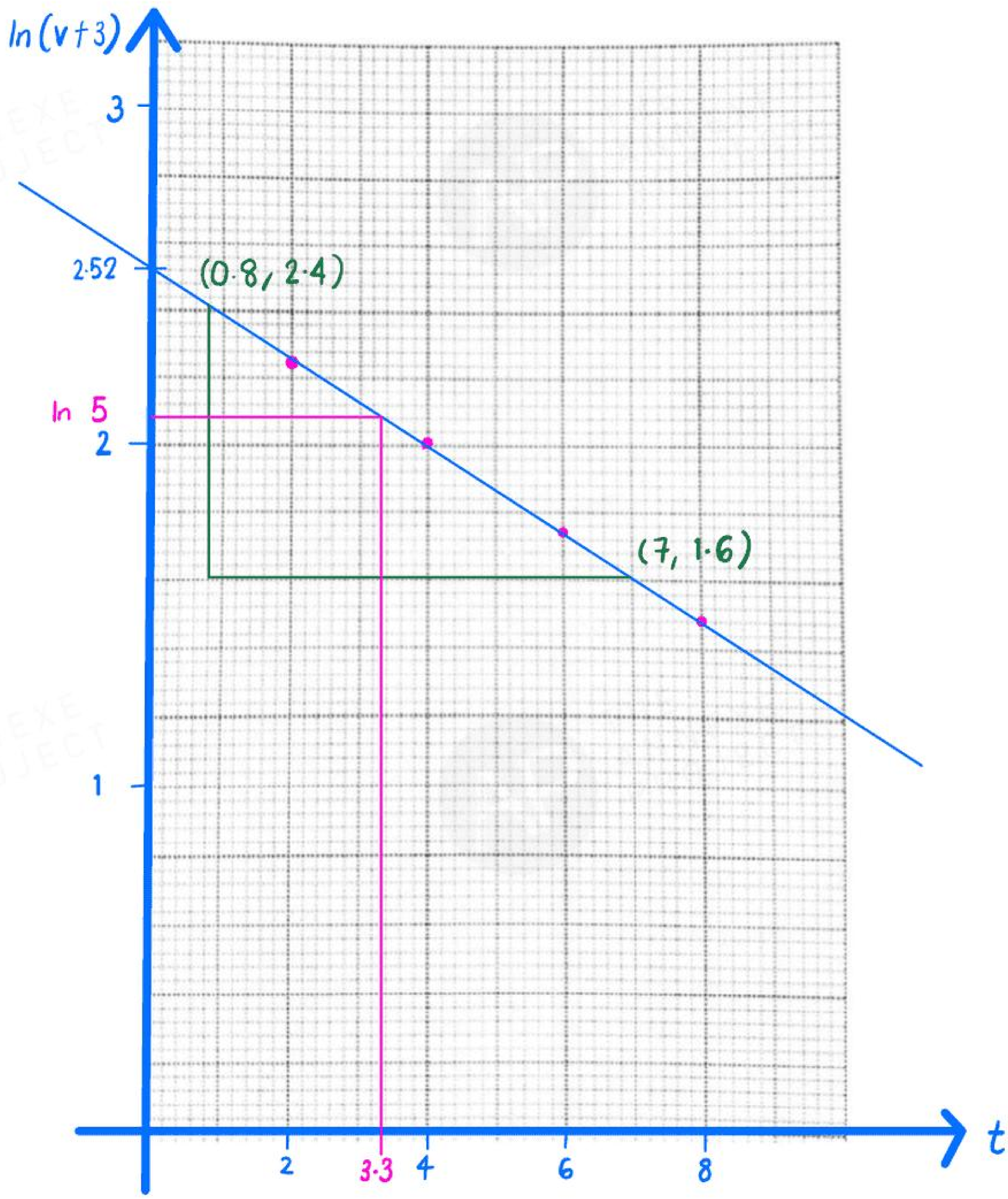
(iii) Hence add a sketch of the graph of $y = |k - 3x|$ to your diagram in **part(i)**. [2]

8 The speed, v m/s, of a vehicle, t s after passing a fixed point O, is given, for $t \geqslant 0$, by $v = Ae^{\frac{t}{k}} - 3$, where A and k are constants. The table below shows corresponding values of t and v.

t	2	4	6	8
v	6.35	4.38	2.67	1.41

(i) Draw the graph of $\ln(v+3)$ plotted against t, using a scale of 1 cm for 1 unit on the t-axis and a scale of 5 cm for 1 unit on the $\ln(v+3)$-axis. [3]

$\ln(v+3)$	2.24	2.00	1.74	1.48



(ii) Use the graph to estimate the value of each of the constants A and k. [5]

$$v = Ae^{\frac{t}{k}} - 3$$
$$v + 3 = Ae^{\frac{t}{k}}$$
$$\ln(v+3) = \ln Ae^{\frac{t}{k}}$$
$$= \ln A + \ln e^{\frac{t}{k}}$$
$$= \ln A + \frac{t}{K}$$
$$= \frac{1}{K}t + \ln A$$

From the graph, $\ln A = 2.52$

$$A = e^{2.52}$$
$$= 12.429$$
$$= \underline{\underline{12.4}}$$

$$\frac{1}{K} = \frac{2.4 - 1.6}{0.8 - 7}$$
$$\frac{1}{K} = \frac{-4}{31}$$
$$K = \frac{-31}{4}$$
$$= \underline{\underline{-7.75}}$$

(iii) State the speed of the vehicle at O. [1]

when $t = 0\,s$, $v = 12.4e^{\frac{0}{-7.75}} - 3$

$$= \underline{\underline{9.4 \text{ m/s}}}$$

(iv) Explain how the graph could be used to find the value of t when the speed of the vehicle is 5 m/s. [1]

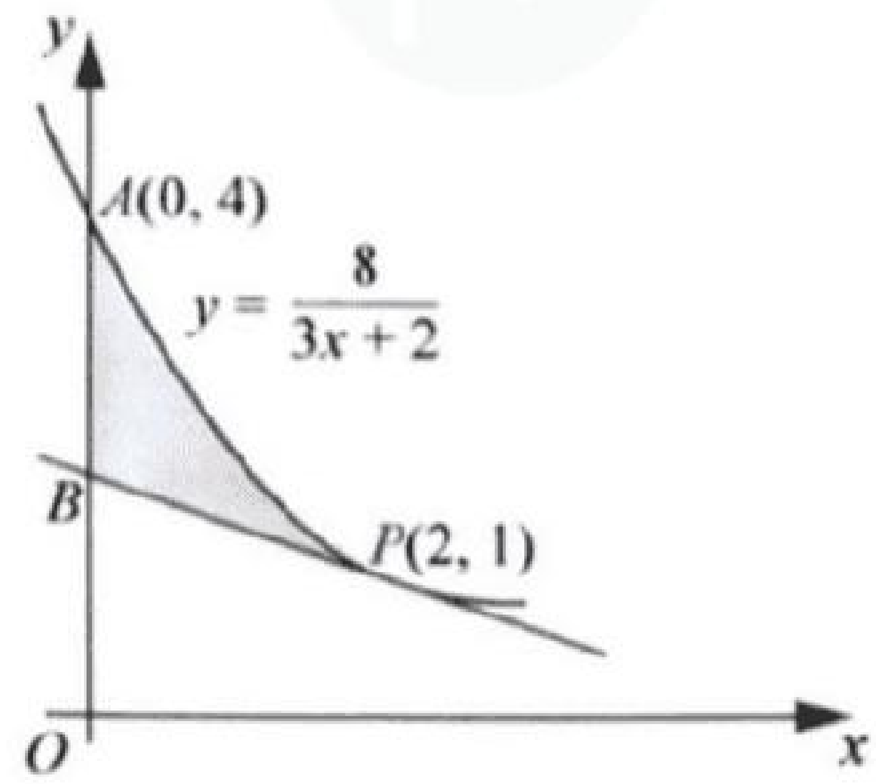
When $v = \ln(5+3) = \ln 8$, we obtain $t = 3.3\,s$ from the graph.

(v) Using your values of A and k, calculate the value of t when the vehicle comes to rest. [1]

Let $v = 0$, $12.4e^{\frac{t}{-7.75}} - 3 = 0$

$$e^{\frac{t}{-7.75}} = \frac{3}{12.4}$$
$$\frac{t}{-7.75} = \ln\left(\frac{3}{12.4}\right)$$
$$\therefore t = \underline{11.0\,s}$$

9



The diagram shows part of the curve $y = \frac{8}{3x+2}$ intersecting the y-axis at $A(0, 4)$. The tangent to the curve at the point $P(2, 1)$ intersects the y-axis at B. Find the exact area of the shaded region. [11]

$y = 8(3x+2)^{-1}$

$\frac{dy}{dx} = -8(3x+2)^{-2}(3)$

$= \frac{-24}{(3x+2)^2}$

when $x = 2$, $\frac{dy}{dx} = \frac{-24}{64} = -\frac{3}{8}$

Equation of tangent BP:

$y - 1 = -\frac{3}{8}(x-2)$

$y - 1 = -\frac{3}{8}x + \frac{3}{4}$

$y = -\frac{3}{8}x + \frac{7}{4}$

Exact area $= \int_0^2 \frac{8}{3x+2} - \left(-\frac{3}{8}x + \frac{7}{4}\right) dx$

$= \left[\frac{8}{3}\ln(3x+2) + \frac{3}{8}\left(\frac{x^2}{2}\right) - \frac{7}{4}x\right]_0^2$

$= \left[\frac{8}{3}\ln 8 + \frac{3}{16}(4) - \frac{7}{4}(2)\right] - \left[\frac{8}{3}\ln 2\right]$

$= \frac{8}{3}\ln 4 + \frac{3}{4} - \frac{7}{2}$

$= \left(\frac{8}{3}\ln 4 - \frac{11}{4}\right)$ sq. units.

10 The roots of the quadratic equation $x^2 = 3x - 4$ are α and β.

(i) Show that $\alpha^3 = 5\alpha - 12$. [3]

$x^2 - 3x + 4 = 0$

Sub α into the above equation:

$\alpha^2 - 3\alpha + 4 = 0$

$\alpha^2 = 3\alpha - 4$ —— (1)

Multiply both sides by α:

$\alpha^3 = \alpha(3\alpha - 4)$

$= 3\alpha^2 - 4\alpha$ —— (2)

Sub (1) into (2):

$\therefore \alpha^3 = 3(3\alpha - 4) - 4\alpha$

$= 5\alpha - 12$ (shown).

(ii) Write down a similar expression for β^3. [1]

$\beta^3 = 5\beta - 12$

(iii) Hence, or otherwise, find the value of $\alpha^3 + \beta^3$. [3]

$\alpha^3 + \beta^3 = (5\alpha - 12) + (5\beta - 12)$

$= 5\alpha + 5\beta - 24 = 5(\alpha + \beta) - 24 = 15 - 24$

$= -9$

Sum of roots: $\alpha + \beta = 3$

Pdt. of roots: $\alpha\beta = 4$

(iv) Find a quadratic equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$. [4]

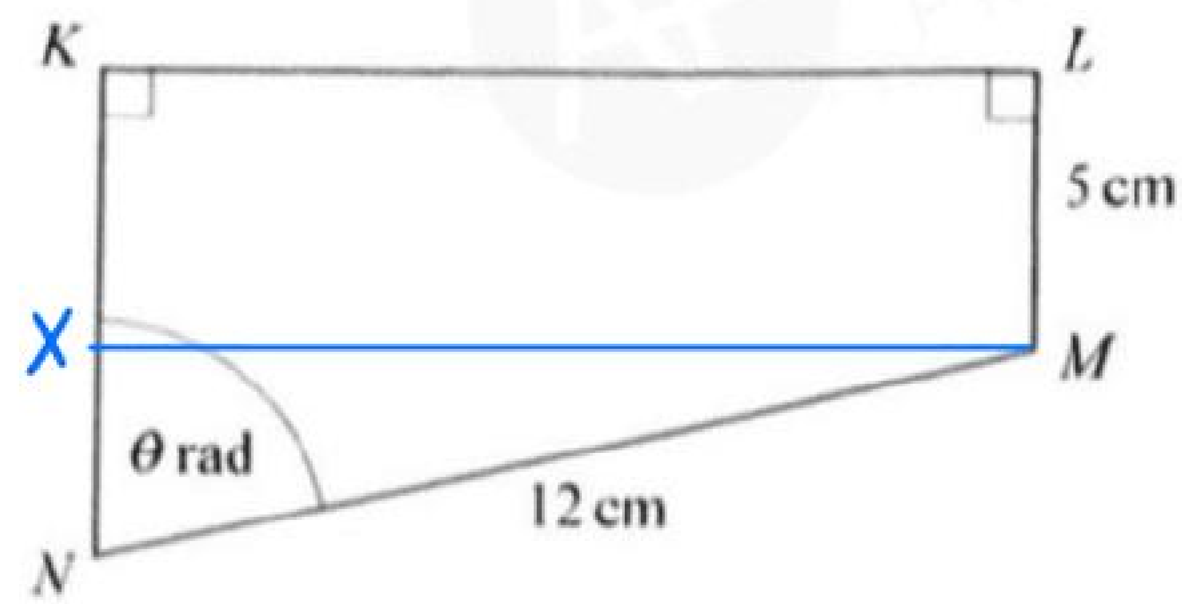
Sum of roots: $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{-9}{4}$ //

Pdt. of roots: $\frac{\alpha^2}{\beta}\left(\frac{\beta^2}{\alpha}\right) = \alpha\beta = 4$ //

Quadratic Equation: $x^2 - \left(\frac{-9}{4}\right)x + 4 = 0$

$$\underline{\underline{4x^2 + 9x + 16 = 0}}$$

11



The diagram shows a metal plate in the shape of a trapezium in which angles NKL and KLM are right angles. The lengths of LM and MN are 5 cm and 12 cm respectively. The acute angle KNM is θ radians.

(i) Show that the perimeter, P cm, is given by $P = 22 + 12\cos\theta + 12\sin\theta$. [2]

$\cos\theta = \frac{XN}{12}$ $\quad\quad$ $\sin\theta = \frac{XM}{12}$

$XN = 12\cos\theta$ $\quad\quad$ $\therefore XM = 12\sin\theta$

$\therefore KN = 5 + 12\cos\theta$

$$\begin{aligned} \Rightarrow P &= KN + KL + 5 + 12 \\ &= 5 + 12\cos\theta + 12\sin\theta + 17 \\ &= \underline{\underline{22 + 12\cos\theta + 12\sin\theta}} \text{ (shown).} \end{aligned}$$

(ii) Find the value of R when $12\cos\theta + 12\sin\theta$ is expressed as $R\cos(\theta - \alpha)$, where R and α are constants, and hence state the maximum perimeter of the plate. [3]

$$\begin{aligned} \text{Let } 12\cos\theta + 12\sin\theta &= R\cos(\theta - \alpha) \\ &= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \end{aligned}$$

Comparing Coefficients,

$R\cos\alpha = 12$ —— (1)

$R\sin\alpha = 12$ —— (2)

$\frac{(2)}{(1)}$: $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$$\begin{aligned} R &= \sqrt{12^2 + 12^2} \\ &= \underline{\underline{12\sqrt{2}}} \end{aligned}$$

$\therefore 12\cos\theta + 12\sin\theta = 12\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

Max. Perimeter $= \underline{\underline{(22 + 12\sqrt{2}) \text{ cm}}}$

(iii) Show that the area of the plate, $A\text{ cm}^2$, is given by $A = 60\sin\theta + 36\sin 2\theta$. [2]

$$A = (12\sin\theta)(5) + \tfrac{1}{2}(12\cos\theta)(12\sin\theta)$$
$$= 60\sin\theta + 72\sin\theta\cos\theta$$
$$= 60\sin\theta + 36(2\sin\theta\cos\theta)$$
$$= \underline{\underline{60\sin\theta + 36\sin 2\theta}} \quad \text{(shown).}$$

(iv) The maximum area of the plate is obtained when the value of θ, which can vary, gives a stationary value of A. Find this value of θ. [5]

$$\frac{dA}{d\theta} = 60\cos\theta + 36\cos 2\theta\,(2)$$
$$= 60\cos\theta + 72\cos 2\theta$$

Let $\frac{dA}{d\theta} = 0$

$$60\cos\theta + 72\cos 2\theta = 0$$
$$60\cos\theta + 72(2\cos^2\theta - 1) = 0$$
$$144\cos^2\theta + 60\cos\theta - 72 = 0$$
$$12\cos^2\theta + 5\cos\theta - 6 = 0$$

$\cos\theta = -0.94549$ (Rej. as θ is acute) or $\cos\theta = 0.52883$

$\underline{\underline{\theta = 1.01 \text{ radians}}}$