MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME	

CENTRE NUMBER	S					INDEX NUMBER				

MATHEMATICS

9758/01

Paper 1

October/November 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

[Turn over

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1 **Do not use a calculator in answering this question.**

The complex numbers z and w satisfy the following equations.

$$iz+2w=-1$$

$$(2-i)z+iw=6$$

Find z and w, giving your answers in the form $a+ib$ where a and b are real numbers. [4]

$iz+2w=-1$ —— 1 $(2-i)z+iw=6$ —— 2

$w = \frac{-1-iz}{2}$ —— 3

Sub 3 into 2:

$(2-i)z + i\left(\frac{-1-iz}{2}\right) = 6$

$(4-2i)z + i(-1-iz) = 12$

$4z - 2iz - i + z = 12$

$(5-2i)z = 12+i$

$z = \frac{12+i}{5-2i}$

$= \frac{12+i}{5-2i} \times \frac{5+2i}{5+2i}$

$= \frac{60+24i+5i-2}{25+4}$

$= \frac{58+29i}{29}$

$= \underline{2+i}$

3: $w = \frac{-1-i(2+i)}{2}$

$= \frac{-1-2i+1}{2}$

$= \underline{-i}$

2 It is given that $f(x) = \tan^{-1}(\sqrt{2}+x)$.

(a) Find $f'(x)$ and $f''(x)$. [3]

(b) Hence find the first three terms of the Maclaurin series for $f(x)$. Give the coefficients correct to 3 significant figures. [3]

(a) $f'(x) = \dfrac{1}{1+(\sqrt{2}+x)^2} = \dfrac{1}{3+2\sqrt{2}x+x^2} = (3+2\sqrt{2}x+x^2)^{-1}$

$f''(x) = -(3+2\sqrt{2}x+x^2)^{-2}\cdot(2\sqrt{2}+2x)$

$= \dfrac{-(2\sqrt{2}+2x)}{(3+2\sqrt{2}x+x^2)^2}$

(b) When $x = 0$:

$f(0) = \tan^{-1}\sqrt{2}$

$f'(0) = \frac{1}{3}$

$f''(0) = \frac{-2\sqrt{2}}{9}$

$f(x) = \tan^{-1}\sqrt{2} + \frac{1}{3}x - \frac{2\sqrt{2}}{9}\frac{x^2}{2!} + \cdots$

$= \tan^{-1}\sqrt{2} + \frac{1}{3}x - \frac{\sqrt{2}}{9}x^2 + \cdots$

$= 0.955 + 0.333x - 0.157x^2 + \cdots$

3 The parametric equations of a curve are $x=\frac{1}{2}(e^{3t}+2e^{-3t})$ and $y=\frac{1}{2}(e^{3t}-2e^{-3t})$.

(a) Using calculus, find the gradient of the normal to the curve at the point where $t=\frac{1}{3}\ln 2$. [3]

(b) By considering x^2 and y^2 or otherwise, find the cartesian equation of the curve, stating any restriction on the values of x. [3]

(a) $\frac{dx}{dt} = \frac{1}{2}e^{3t}\cdot 3 + e^{-3t}\cdot(-3)$

$= \frac{3}{2}e^{3t} - 3e^{-3t} = \frac{1}{2}e^{-3t}(3e^{6t}-6)$

$\frac{dy}{dt} = \frac{1}{2}e^{3t}\cdot 3 - e^{-3t}\cdot(-3)$

$= \frac{3}{2}e^{3t} + 3e^{-3t} = \frac{1}{2}e^{-3t}(3e^{6t}+6)$

$\frac{dy}{dx} = \frac{\cancel{\frac{1}{2}e^{-3t}}(3e^{6t}+6)}{\cancel{\frac{1}{2}e^{-3t}}(3e^{6t}-6)} = \frac{3e^{6t}+6}{3e^{6t}-6}$

When $t=\frac{1}{3}\ln 2$, $\frac{dy}{dx} = \frac{3e^{2\ln 2}+6}{3e^{2\ln 2}-6}$

$= \frac{3e^{\ln 4}+6}{3e^{\ln 4}-6}$

$= \frac{3(4)+6}{3(4)-6} = \frac{18}{6} = 3$

Hence, gradient of normal $= -\frac{1}{3}$

For perpendicular lines,
$m_1 \times m_2 = -1$

(b) $x^2 = \frac{1}{4}(e^{6t} + 2e^{3t}(2e^{-3t}) + 4e^{-6t})$

$= \frac{1}{4}e^{6t} + 1 + e^{-6t}$

$y^2 = \frac{1}{4}(e^{6t} - 2e^{3t}(2e^{-3t}) + 4e^{-6t})$

$= \frac{1}{4}e^{6t} - 1 + e^{-6t}$

$\therefore x^2 - y^2 = (\frac{1}{4}e^{6t} + 1 + e^{-6t}) - (\frac{1}{4}e^{6t} - 1 + e^{-6t})$

$x^2 - y^2 = 2$

$x^2 = y^2 + 2$

For all values of y, $x^2 \geq 2$

$$x^2 - \sqrt{2}^2 \geq 0$$

$$(x - \sqrt{2})(x + \sqrt{2}) \geq 0$$

$$\therefore x \leq -\sqrt{2} \quad \text{or} \quad x \geq \sqrt{2}$$

Since $e^{3t} > 0$ and $e^{-3t} > 0$ for all real values of t,

$$x = \frac{1}{2}(e^{3t} + 2e^{-3t}) > 0$$

Hence, $x \leq -\sqrt{2}$ is rejected and the restriction on x is such that $\underline{x \geq \sqrt{2}}$.

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4 (a) Show that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. [2]

(b) Show that $\sin 2x \tan x = 2\sin^2 x$. [1]

(c) Hence, find the exact value of $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \cot 3x \, dx$. [4]

ⓐ
$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x \text{ (shown).}$$

ⓑ
$$\sin 2x \tan x = 2\sin x \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}}$$
$$= 2\sin^2 x \text{ (shown)}$$

ⓒ
$$\int \operatorname{cosec} 6x \cdot \cot 3x \, dx$$
$$= \int \frac{1}{\sin 6x \tan 3x} \, dx$$
$$= \int \frac{1}{2\sin^2 3x} \, dx$$
$$= \frac{1}{2}\int \operatorname{cosec}^2 3x \, dx$$
$$= -\frac{1}{2}\int -\operatorname{cosec}^2 3x \, dx$$
$$= -\frac{1}{2}\left[\frac{\cot 3x}{3}\right] + c$$
$$= -\frac{1}{6}\cot 3x + C$$

$$\therefore \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \cot 3x \, dx = \left[-\frac{1}{6}\cot 3x\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$$
$$= \left[-\frac{1}{6}\cot\frac{\pi}{3} + \frac{1}{6}\cot\frac{\pi}{6}\right]$$
$$= \left[-\frac{1}{6}\cdot\frac{1}{\tan\frac{\pi}{3}} + \frac{1}{6}\cdot\frac{1}{\tan\frac{\pi}{6}}\right]$$
$$= -\frac{1}{6}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{6}(\sqrt{3})$$
$$= \frac{-\sqrt{3}}{18} + \frac{3\sqrt{3}}{18} = \frac{\sqrt{3}}{9}$$

5 The line with equation $y = mx$ is a tangent to the curve with equation

$$(x+8)^2 + (y-14)^2 = 52.$$

(a) Show that m satisfies the equation

$$3m^2 + 56m + 36 = 0.$$ [4]

A and B are points on the curve. The tangent at A and the tangent at B intersect at the origin.

(b) Find the coordinates of A and B. [4]

(a) $y = mx$ —— (1)

$(x+8)^2 + (y-14)^2 = 52$ —— (2)

Sub (1) into (2):

$x^2 + 16x + 64 + (mx - 14)^2 = 52$

$x^2 + 16x + 64 + m^2x^2 - 28mx + 196 - 52 = 0$

$(1+m^2)x^2 + (16-28m)x + 208 = 0$

$b^2 - 4ac = 0$ (since $y = mx$ is a tangent to the curve)

$(16-28m)^2 - 4(1+m^2)(208) = 0$

$256 - 896m + 784m^2 - 832 - 832m^2 = 0$

$-48m^2 - 896m - 576 = 0$

$3m^2 + 56m + 36 = 0$ (shown).

(b) Solving $3m^2 + 56m + 36 = 0$

$(3m+2)(m+18) = 0$

$m = -\frac{2}{3}$ or $m = -18$

When $m = -\frac{2}{3}$:

$(1 + \frac{4}{9})x^2 + (16 + \frac{56}{3})x + 208 = 0$

$\frac{13}{9}x^2 + \frac{104}{3}x + 208 = 0$

By GC, $x = -12$

When $x = -12$, $y = -\frac{2}{3}(-12) = 8$

$\therefore A = (-12, 8)$

When $m = -18$:

$(1 + 324)x^2 + (16 + 504)x + 208 = 0$

$325x^2 + 520x + 208 = 0$

By GC, $x = -\frac{4}{5}$

When $x = -\frac{4}{5}$, $y = -18(-\frac{4}{5}) = \frac{72}{5}$

$\therefore B = (-\frac{4}{5}, \frac{72}{5})$

6 The function f is defined by

$$f: x \mapsto \frac{ax+k}{x-a}, \quad x \in \mathbb{R}, \quad x \neq a$$

where a and k are constants.

(a) Describe fully a sequence of transformations which transforms the curve $y = \frac{1}{x}$ onto the curve $y = f(x)$. [4]

(b) Find $f^{-1}(x)$. [2]

(c) Hence, or otherwise, find $f^2(x)$. [1]

(d) Find $f^{2023}(1)$ in terms of a and k. [2]

(a) Step 1:

$$\begin{array}{r} a \\ x-a \overline{\smash{)}\, ax+K} \\ -(ax-a^2) \\ \hline K+a^2 \\ \hline \end{array}$$

$$\therefore f(x) = \frac{ax+K}{x-a} = a + \frac{K+a^2}{x-a}$$

$$\boxed{y = \frac{1}{x} \xrightarrow[x-a]{\text{replace } x \text{ with}} y = \frac{1}{x-a} \xrightarrow[\frac{y}{K+a^2}]{\text{replace } y \text{ with}} y = \frac{K+a^2}{x-a} \xrightarrow[y-a]{\text{replace } y \text{ with}} y = a + \frac{K+a^2}{x-a}}$$

1). translation of a units along positive x-axis.

2). scaling of factor $K+a^2$ parallel to the y-axis.

3). translation of a units along positive y-axis.

(b) $y = \frac{ax+K}{x-a}$

$y(x-a) = ax+K$

$yx - ax = ay + K$

$x(y-a) = ay+K$

$x = \frac{ay+K}{y-a}$

$f^{-1}(x) = \frac{ax+K}{x-a}, \quad x \in \mathbb{R}, \ x \neq a.$

(c) $f(x) = f^{-1}(x)$

$f^2(x) = ff^{-1}(x)$

$= x$

(d) $f^{2023}(1) = f(1)$

$= \frac{a+K}{1-a}$

Because $f(x) = f^{-1}(x)$ and $D_f = D_{f^{-1}}$, $f(x)$ is a self-inverse function.

For self-inverse function, $f^{odd}(x) = f(x)$ and $f^{even}(x) = x$

7 A curve C has equation $y = x^{-3}\ln x$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{1-3\ln x}{x^4}$ and hence find the coordinates of the turning point of C. [4]

(b) Find the exact area enclosed by C, the x-axis and the line $x = 3$. [5]

ⓐ Applying Product Rule:

$$\frac{dy}{dx} = x^{-3}\left(\frac{1}{x}\right) + \ln x\left(-3x^{-4}\right)$$

$$= \frac{1}{x^4} - \frac{3\ln x}{x^4}$$

$$= \frac{1-3\ln x}{x^4} \text{ (shown)}$$

Let $\frac{dy}{dx} = 0$: $1 - 3\ln x = 0$

$\ln x = \frac{1}{3}$

$x = e^{\frac{1}{3}}$

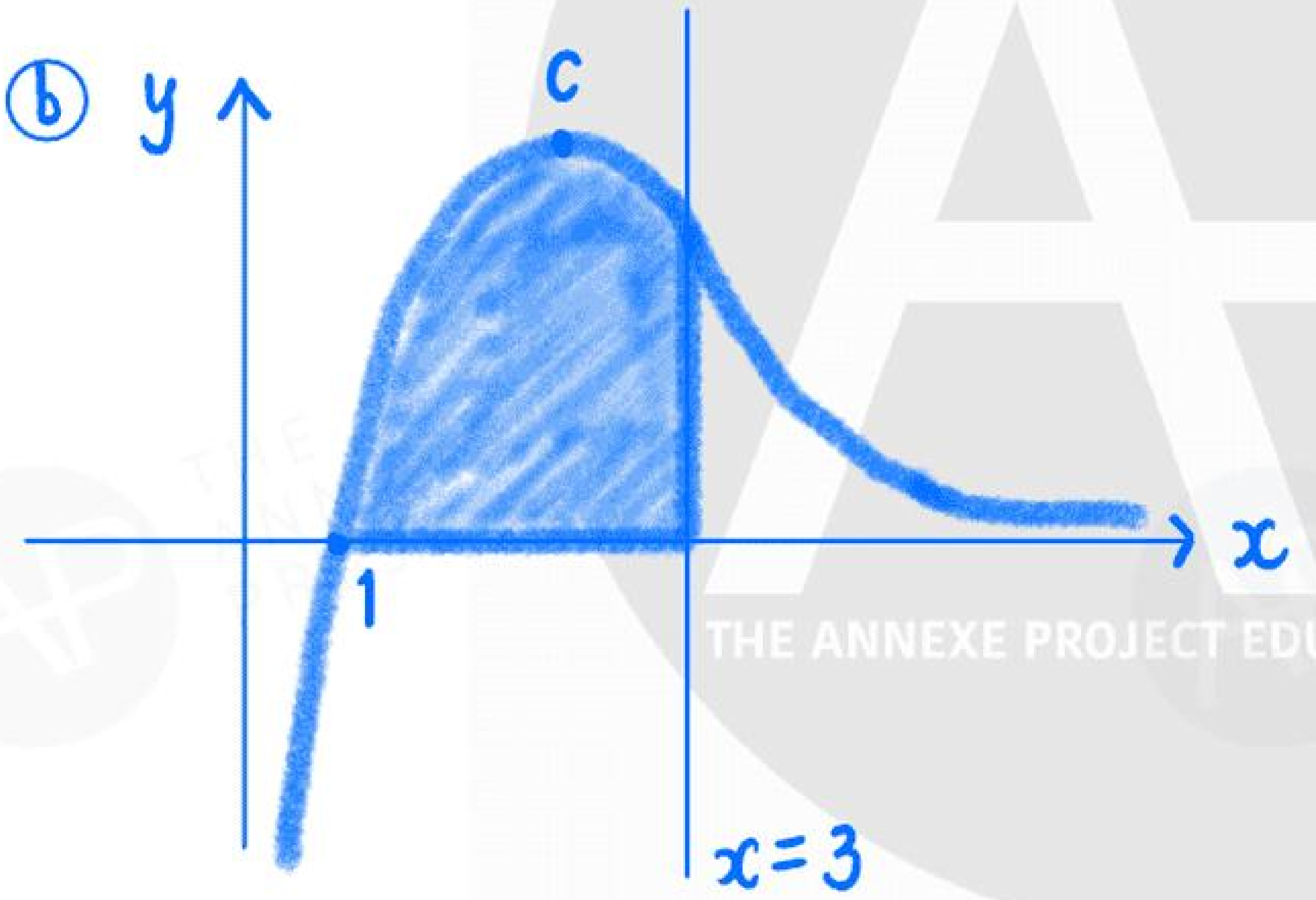
When $x = e^{\frac{1}{3}}$, $y = \left(e^{\frac{1}{3}}\right)^{-3} \ln e^{\frac{1}{3}}$

$= e^{-1} \cdot \frac{1}{3}$

$= \frac{1}{3e}$

$\therefore C = \left(e^{\frac{1}{3}}, \frac{1}{3e}\right)$

ⓑ



$u = \ln x$ | $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ | $v = \frac{x^{-2}}{-2}$

$= -\frac{1}{2x^2}$

Applying integration by Parts:

$$A = \int_1^3 x^{-3}\ln x \, dx = \left[\frac{-1}{2x^2} \cdot \ln x\right]_1^3 - \int_1^3 \frac{-1}{2x^2} \cdot \frac{1}{x} \, dx$$

$$= \left(\frac{-1}{18}\ln 3\right) + \frac{1}{2}\int_1^3 x^{-3} \, dx$$

$$= \frac{-1}{18}\ln 3 + \frac{1}{2}\left[\frac{x^{-2}}{-2}\right]_1^3$$

$$= \frac{-1}{18}\ln 3 - \frac{1}{4}\left[\frac{1}{3^2} - \frac{1}{1^2}\right]$$

$$= \frac{-1}{18}\ln 3 - \frac{1}{4}\left(-\frac{8}{9}\right)$$

$$= \left(\frac{-1}{18}\ln 3 + \frac{2}{9}\right) \text{ units}^2$$

8 (a) Find $\displaystyle\int \frac{2x-1}{x^2+2x+1}\,dx$. [4]

(b) Find the exact value of $\displaystyle\int_0^2 \frac{|2x-1|}{x^2+2x+1}\,dx$. [3]

(a)

$$\int \frac{2x-1}{x^2+2x+1}\,dx = \int \frac{(2x+2)-3}{x^2+2x+1}\,dx$$

$$= \int \frac{2x+2}{x^2+2x+1}\,dx - 3\int \frac{1}{x^2+2x+1}\,dx$$

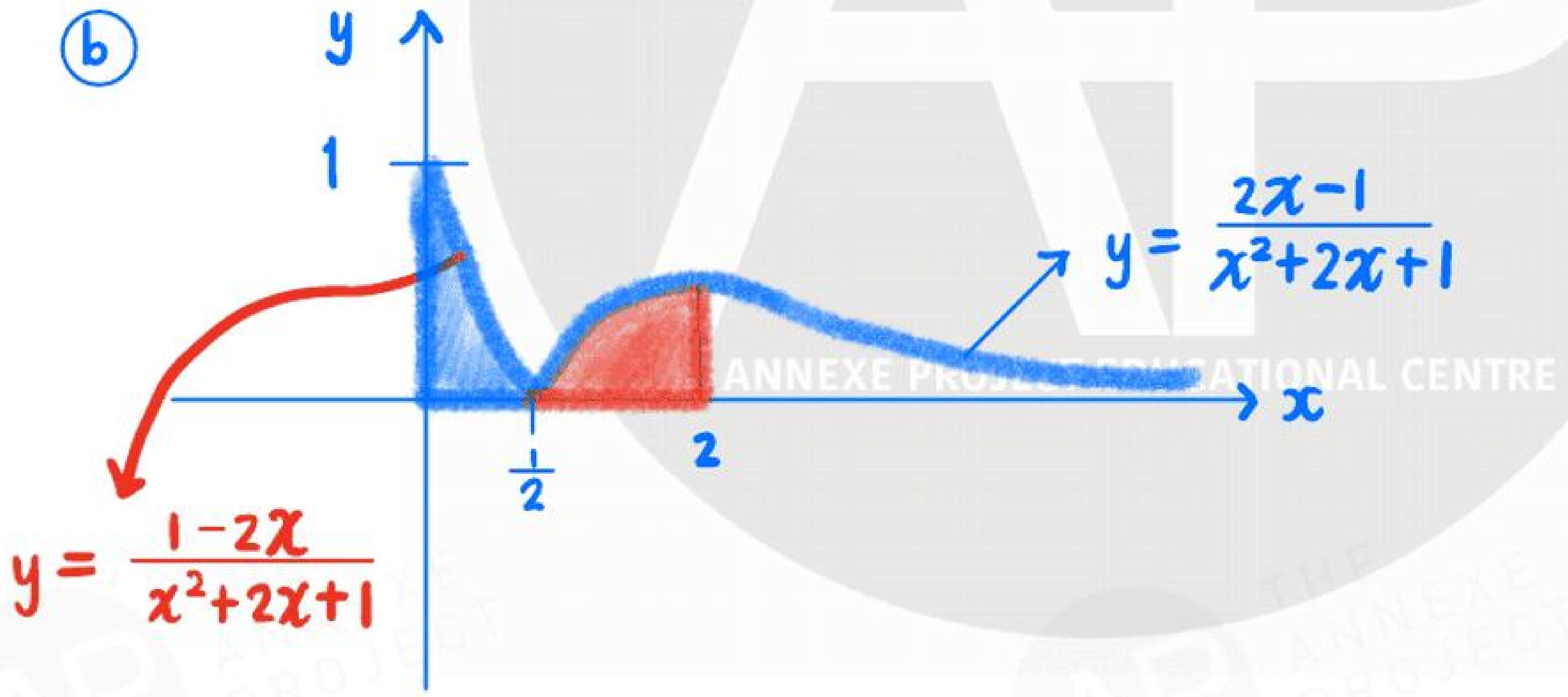
$$= \ln(x^2+2x+1) - 3\int (x+1)^{-2}\,dx$$

$$= \ln(x^2+2x+1) - 3\left[\frac{(x+1)^{-1}}{-1}\right] + c$$

$$= \ln(x^2+2x+1) + \frac{3}{x+1} + c$$

$$\boxed{\begin{array}{l} x^2+2x+1 = (x+1)^2 \\ \text{hence for all } x \in \mathbb{R},\ x \neq -1, \\ x^2+2x+1 > 0 \end{array}}$$

(b)



$$\int_0^2 \frac{|2x-1|}{x^2+2x+1}\,dx = \int_0^{\frac{1}{2}} \frac{1-2x}{x^2+2x+1}\,dx + \int_{\frac{1}{2}}^2 \frac{2x-1}{x^2+2x+1}\,dx$$

$$= -\int_0^{\frac{1}{2}} \frac{2x-1}{x^2+2x+1}\,dx + \int_{\frac{1}{2}}^2 \frac{2x-1}{x^2+2x+1}\,dx$$

$$= -\left[\ln(x^2+2x+1) + \frac{3}{x+1}\right]_0^{\frac{1}{2}} + \left[\ln(x^2+2x+1) + \frac{3}{x+1}\right]_{\frac{1}{2}}^2$$

$$= -\left[(\ln 2.25 + 2) - (0+3)\right] + \left[(\ln 9 + 1) - (\ln 2.25 + 2)\right]$$

$$= \ln 9 - 2\ln 2.25$$

$$= \ln 9 - \ln \frac{81}{16}$$

$$= \ln \frac{9}{\left(\frac{81}{16}\right)}$$

$$= \left(\ln \frac{16}{9}\right) \text{ units}^2$$

9 (a) An arithmetic series has first term a and common difference d, where $d \neq 0$. The first, third and fifteenth terms of this series are the first, second and third terms of a geometric series. Find d in terms of a. [3]

(b) A geometric series has first term $\sin\theta$ and common ratio $-\cos\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that the sum to infinity of this series is $\tan k\theta$, where k is a constant to be found. [3]

(ii) Given that $\theta = \frac{\pi}{3}$, find the exact sum of the first seven terms of this series. [2]

(a) GP : a , $a+2d$, $a+14d$, ...

common ratio: $\frac{a+2d}{a} = \frac{a+14d}{a+2d}$

$$(a+2d)^2 = a(a+14d)$$

$$\cancel{a^2} + 4ad + 4d^2 = \cancel{a^2} + 14ad$$

$$4d^2 - 10ad = 0$$

$$2d(2d - 5a) = 0$$

$d = 0$ (rej.) or $d = \frac{5}{2}a$

(b) i. $S_\infty = \frac{a}{1-r}$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+(2\cos^2\frac{\theta}{2}-1)}$$

$$= \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$= \tan\frac{\theta}{2}$ where $k = \frac{1}{2}$

ii. first term $= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

common ratio $r = -\cos\frac{\pi}{3} = -\frac{1}{2}$

$$S_7 = \frac{\frac{\sqrt{3}}{2}\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left(1+\frac{1}{128}\right)}{\frac{3}{2}}$$

$$= \frac{\sqrt{3}}{3}\left(\frac{129}{128}\right)$$

$$= \frac{43\sqrt{3}}{128}$$

10 A curve C has equation $y = ax + b + \dfrac{a+2b}{x-1}$, where a and b are real constants such that $a > 0$, $b \neq -\frac{1}{2}a$ and $x \neq 1$.

(a) Given that C has no stationary points, use differentiation to find the relationship between a and b. [3]

$\frac{dy}{dx} = a + (a+2b)(-1)(x-1)^{-2}$

$= a - \dfrac{a+2b}{(x-1)^2}$

Let $\frac{dy}{dx} = 0$: $\quad \dfrac{a+2b}{(x-1)^2} = a$

$a + 2b = a(x^2 - 2x + 1)$

$ax^2 - 2ax - 2b = 0$

Since C has no stationary points, $b^2 - 4ac < 0$

$(-2a)^2 - 4a(-2b) < 0$

$4a^2 + 8ab < 0$

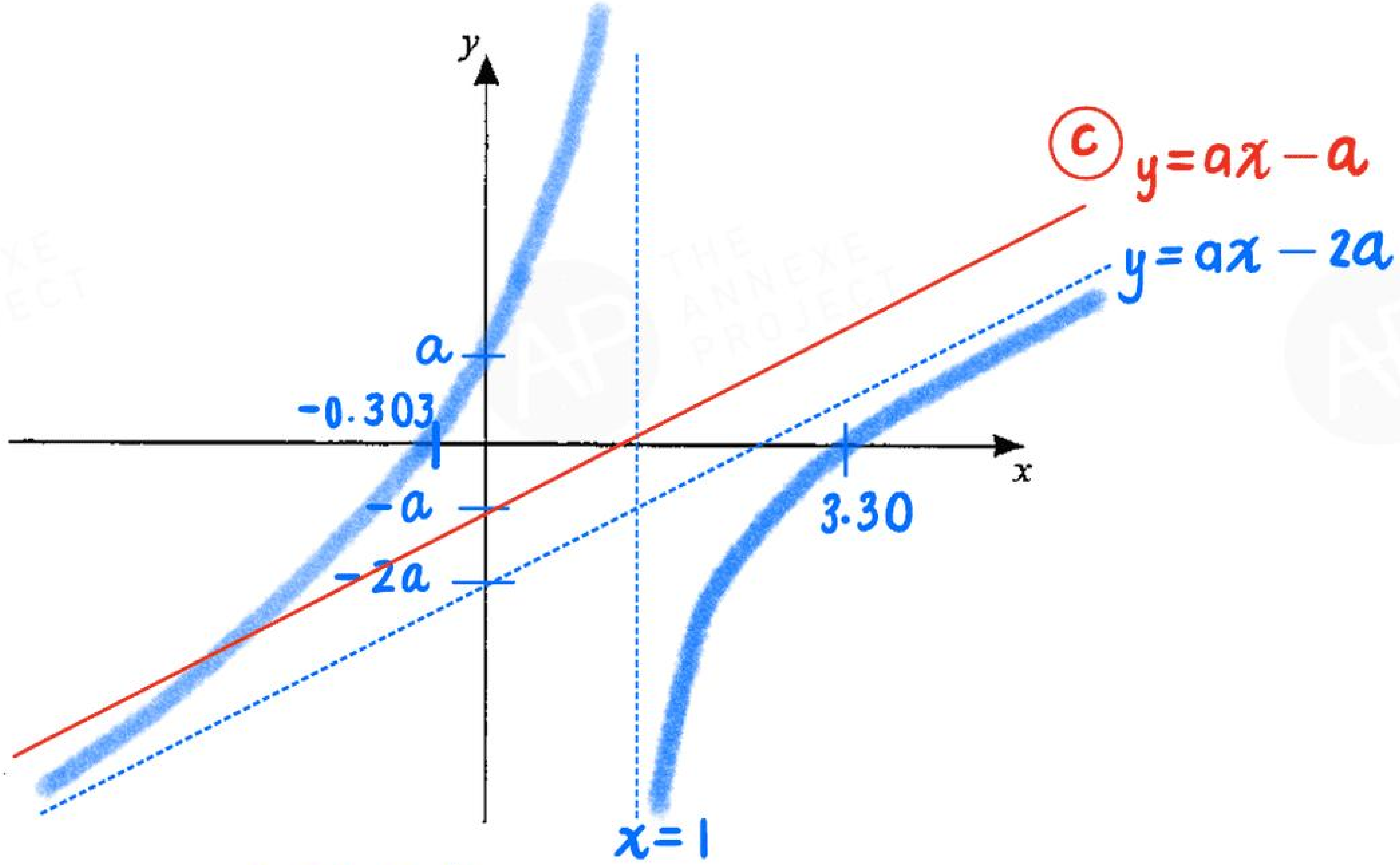
$4a(a + 2b) < 0$

Given $a > 0$, then $a + 2b < 0$

$a < -2b$

It is now given that $b = -2a$.

(b) Sketch C on the axes on page 19 stating the equations of any asymptotes and the coordinates of the points where C crosses the axes. [4]



$y = ax - 2a + \dfrac{a + 2(-2a)}{x-1}$

$= ax - 2a - \dfrac{3a}{x-1}$

vertical asymptote: $x = 1$

oblique asymptote: $y = ax - 2a$

When $x = 0$, $y = -2a + 3a$

$y = a$

When $y = 0$,

$ax - 2a = \dfrac{3a}{x-1}$

$x - 2 = \dfrac{3}{x-1}$

$$(x-2)(x-1) = 3$$
$$x^2 - 3x - 1 = 0$$
By GC:
$x = -0.303$ or 3.30

(c) On the same axes, sketch the graph of $y = ax - a$. [1]

(d) Hence solve the inequality $x - 2 - \frac{3}{x-1} \leqslant x - 1$. [2]

(d) consider $a = 1$:

by GC: the graphs of $y = x - 2 - \frac{3}{x-1}$ and $y = x - 1$ meet at $x = -2$.

from the graphs above, $x \leq -2$ or $x > 1$ for $x - 2 - \frac{3}{x-1} \leq x - 1$.

11 A gas company has plans to install a pipeline from a gas field to a storage facility. One part of the route for the pipeline has to pass under a river. This part of the pipeline is in a straight line between two points, P and Q.

Points are defined relative to an origin $(0, 0, 0)$ at the gas field. The x-, y- and z-axes are in the directions east, north and vertically upwards respectively, with units in metres. P has coordinates $(1136, 92, p)$ and Q has coordinates $(200, 20, -15)$.

(a) The length of the pipeline PQ is 939 m. Given that the level of P is below that of Q, find the value of p. [3]

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} - \begin{pmatrix} 1136 \\ 92 \\ p \end{pmatrix}$$

$$= \begin{pmatrix} -936 \\ -72 \\ -15-p \end{pmatrix}$$

$$|\overrightarrow{PQ}| = 939$$

$$\therefore \sqrt{(-936)^2 + (-72)^2 + (-15-p)^2} = 939$$

$$876096 + 5184 + 225 + 30p + p^2 = 881721$$

$$p^2 + 30p - 216 = 0$$

$$(p-6)(p+36) = 0$$

$$p = 6 \text{ (rej.)} \qquad \underline{p = -36}$$

A thin layer of rock lies below the ground. This layer is modelled as a plane. Three points in this plane are $(400, 600, -20)$, $(500, 200, -70)$ and $(600, -340, -50)$.

(b) Find the cartesian equation of this plane. [4]

(c) Hence find the coordinates of the point where the pipeline meets the rock. [4]

(d) Find the angle that the pipeline between the points P and Q makes with the horizontal. [2]

(b) Let $\overrightarrow{OA} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 500 \\ 200 \\ -70 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 600 \\ -340 \\ -50 \end{pmatrix}$

$$\overrightarrow{AB} = \begin{pmatrix} 500 \\ 200 \\ -70 \end{pmatrix} - \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} = \begin{pmatrix} 100 \\ -400 \\ -50 \end{pmatrix} = 50\begin{pmatrix} 2 \\ -8 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 600 \\ -340 \\ -50 \end{pmatrix} - \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} = \begin{pmatrix} 200 \\ -940 \\ -30 \end{pmatrix} = 10\begin{pmatrix} 20 \\ -94 \\ -3 \end{pmatrix}$$

$$\underset{\sim}{n} = \begin{pmatrix} 2 \\ -8 \\ -1 \end{pmatrix} \times \begin{pmatrix} 20 \\ -94 \\ -3 \end{pmatrix} = \begin{pmatrix} -70 \\ -14 \\ -28 \end{pmatrix} = -14\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2000 + 600 - 40 = 2560$$

cartesian equation: $\underline{5x + y + 2z = 2560}$

(c) Since $p = -36$, $\overrightarrow{PQ} = \begin{pmatrix} -936 \\ -72 \\ -15+36 \end{pmatrix} = \begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix} = 3\begin{pmatrix} -312 \\ -24 \\ 7 \end{pmatrix}$

$\therefore\ l_{PQ} : \underset{\sim}{r} = \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} -312 \\ -24 \\ 7 \end{pmatrix},\ \lambda \in \mathbb{R}$

<u>Sub eqn. of line PQ into eqn. of plane</u>

$$\begin{pmatrix} 200 - 312\lambda \\ 20 - 24\lambda \\ -15 + 7\lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2560$$

$$1000 - 1560\lambda + 20 - 24\lambda - 30 + 14\lambda = 2560$$

$$-1570\lambda = 1570$$

$$\lambda = -1$$

intersection point $= \begin{pmatrix} 200 - 312(-1) \\ 20 - 24(-1) \\ -15 + 7(-1) \end{pmatrix}$

$= \begin{pmatrix} 512 \\ 44 \\ -22 \end{pmatrix}$

coordinates $= (512, 44, -22)$

(d) Angle between pipeline and horizontal

$$= \sin^{-1} \frac{\begin{pmatrix} -312 \\ -24 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{312^2 + 24^2 + 7^2}}$$

$$= \sin^{-1} \frac{7}{\sqrt{97969}}$$

$= 1.2815$

$= 1.3°$ (1 d.p.)

THE ANNEXE PROJECT
EDUCATIONAL CENTRE
EST^{D} 2008

12 Scientists are interested in the population of a particular species. They attempt to model the population P at time t days using a differential equation. Initially the population is observed to be 50 and after 10 days the population is 100.

The first model the scientists use assumes that the rate of change of the population is proportional to the population.

(a) Write down a differential equation for this model and solve it for P in terms of t. [5]

Let P be the population of the species at any time t (days).

Given $\frac{dP}{dt} = kP$

$$\int \frac{1}{P}\, dP = k\int dt$$

$$\ln P = kt + c$$

$$P = e^{kt+c}$$

$$= e^{kt}\cdot e^{c}$$

$$= Ae^{kt}, \text{ where } A = e^{c}$$

There's no need to write $\ln|P|$ as $P > 0$

When $t = 0$, $P = 50$: $50 = Ae^{0}$

$\therefore A = 50$

When $t = 10$, $P = 100$: $100 = 50e^{10k}$

$$2 = e^{10k}$$

$$\ln 2 = \ln e^{10k}$$

$$10k = \ln 2$$

$$k = \frac{\ln 2}{10}$$

$$\therefore P = 50e^{\frac{\ln 2}{10}t}$$

To allow for constraints on population growth, the model is refined to

$$\frac{dP}{dt} = \lambda P(500 - P)$$

where λ is a constant.

(b) Solve this differential equation to find P in terms of t. [6]

(c) Using the refined model, state the population of this species in the long term. Comment on how this value suggests the refined model is an improvement on the first model. [2]

(b) $$\int \frac{1}{P(500-P)}\, dP = \lambda \int dt$$

$$\frac{1}{500}\int \frac{1}{P} + \frac{1}{500-P}\, dP = \lambda\int dt$$

$$\ln P - \ln|500 - P| = 500\lambda t + c$$

By Partial Fraction

$$\frac{1}{P(500-P)} = \frac{A}{P} + \frac{B}{500-P}$$

$$= \frac{A(500-P) + BP}{P(500-P)}$$

$$\therefore 1 = A(500-P) + BP$$

$$\ln \frac{P}{|500-P|} = 500\lambda t + C$$

$$\frac{P}{|500-P|} = e^{500\lambda t + C}$$

$$\frac{P}{500-P} = \pm e^{c} e^{500\lambda t}$$

$$\frac{P}{500-P} = Be^{500\lambda t}, \quad \text{where } B = \pm e^{c}$$

Sub $P = 500$:

$1 = 500B$

$B = \frac{1}{500}$

Sub $P = 0$:

$1 = 500A$

$A = \frac{1}{500}$

When $t = 0$, $P = 50$:

$$\frac{50}{450} = Be^{0}$$

$$\therefore B = \frac{1}{9}$$

When $t = 10$, $P = 100$:

$$\frac{100}{400} = \frac{1}{9}e^{5000\lambda}$$

$$\frac{9}{4} = e^{5000\lambda}$$

$$\ln\frac{9}{4} = 5000\lambda$$

$$\lambda = \frac{\ln\frac{9}{4}}{5000}$$

Since $\frac{P}{50}$

$$P = (500-P)Be^{500\lambda t}$$

$$P + PBe^{500\lambda t} = 500Be^{500\lambda t}$$

$$P = \frac{500Be^{500\lambda t}}{1 + Be^{500\lambda t}}$$

$$= \frac{500\left(\frac{1}{9}\right)e^{500\left(\frac{1}{5000}\ln\frac{9}{4}\right)t}}{1 + \frac{1}{9}e^{500\left(\frac{1}{5000}\ln\frac{9}{4}\right)t}}$$

$$= \frac{\frac{500}{9}e^{\frac{1}{10}\ln\frac{9}{4}t}}{1 + \frac{1}{9}e^{\frac{1}{10}\ln\frac{9}{4}t}}$$

$$= \frac{500e^{\frac{1}{10}\ln\frac{9}{4}t}}{9 + e^{\frac{1}{10}\ln\frac{9}{4}t}}$$

$$= \frac{500}{9e^{-\frac{1}{10}\ln\frac{9}{4}t} + 1}$$

divide numerator and denominator by $e^{\frac{1}{10}\ln\frac{9}{4}t}$

(c) As $t \to \infty$, $e^{-\frac{1}{10}\ln\frac{9}{4}t} \longrightarrow 0$, $P \longrightarrow 500$

Using the refined model, population approaches 500 in the long run.

In the first model $P = 50e^{\frac{\ln 2}{10}t}$, as $t \longrightarrow \infty$, $P \longrightarrow \infty$.

This does not make sense as population grows indefinitely.