MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS 9758/01

Paper 1

For examination from 2025

SPECIMEN PAPER

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

1 A circular ink-blot is expanding such that the rate of change of its diameter D with respect to time t is $0.3\,\text{cm}\,\text{s}^{-1}$. Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 2 cm. [4]

Given: $\frac{dD}{dt} = 0.3 \text{ cm s}^{-1}$

Find $\frac{dC}{dt}$ and $\frac{dA}{dt}$ when $r = 2$ cm, $D = 4$ cm

Form the chainrule:

$$\frac{dC}{dt} = \frac{dC}{dD} \times \frac{dD}{dt}$$

$$= \pi \times 0.3$$

$$= 0.3\pi \text{ cm s}^{-1}$$

also $C = \pi D$

$\therefore \frac{dC}{dD} = \pi$

Form the chainrule:

$$\frac{dA}{dt} = \frac{dA}{dD} \times \frac{dD}{dt}$$

$$= \frac{\pi}{2}(4) \times 0.3$$

$$= 0.6\pi \text{ cm}^2 \text{ s}^{-1}$$

also $A = \pi r^2$

$= \pi \left(\frac{D}{2}\right)^2$

$= \frac{\pi}{4} D^2$

$\therefore \frac{dA}{dD} = \frac{\pi}{2} D$

2 The curve C with equation $y = x^3$ is transformed onto the curve with equation $y = \mathrm{f}(x)$ by a translation of 2 units in the negative x-direction, followed by a stretch of factor $\frac{1}{4}$ parallel to the y-axis.

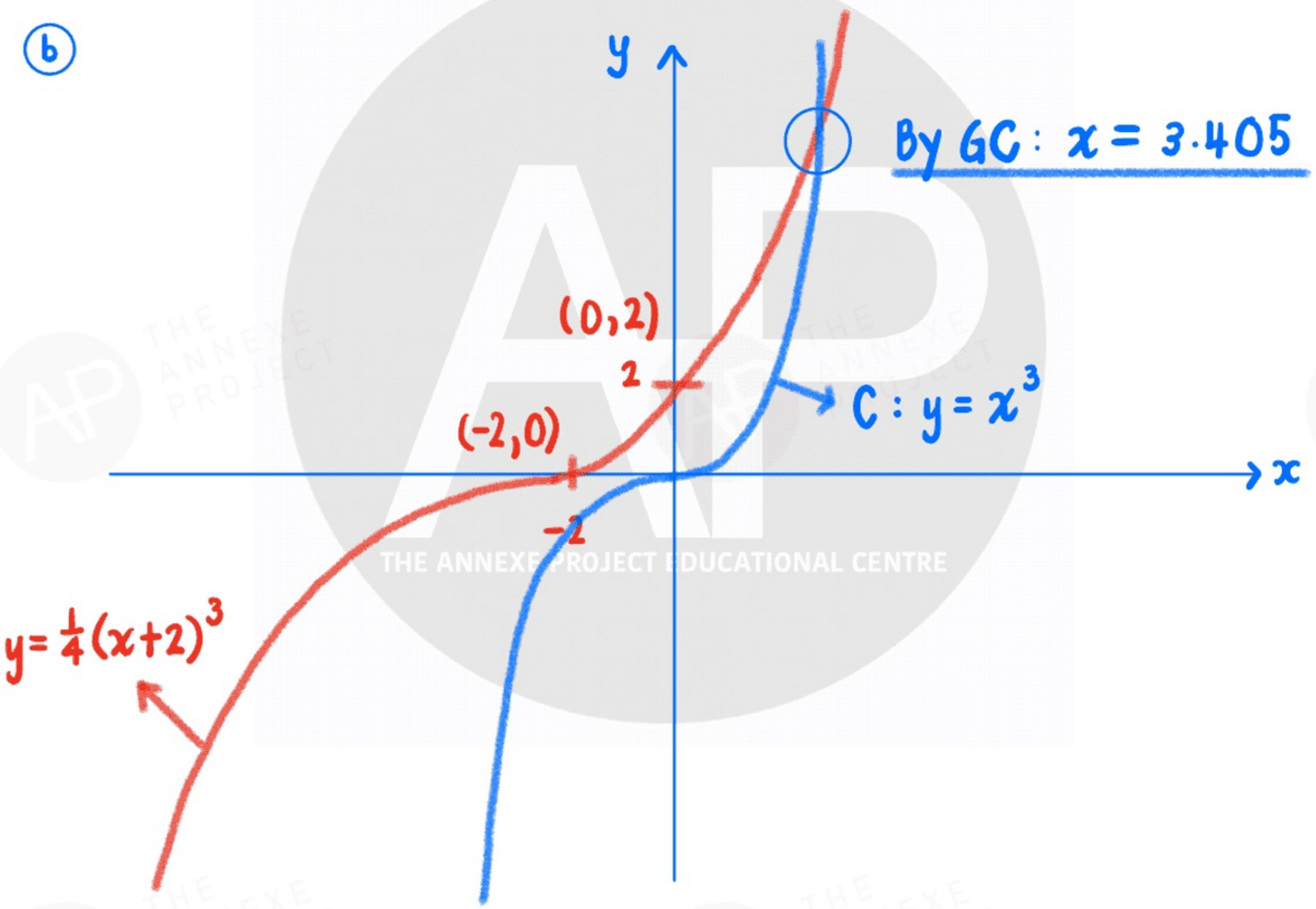
(a) Write down the equation of the new curve. [1]

(b) Sketch C and the curve with equation $y = \mathrm{f}(x)$ on the same diagram, stating the values of the coordinates of the points where $y = \mathrm{f}(x)$ crosses the x- and y-axes. Find the x-coordinate(s) of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

(a) $f(x) = x^3 \xrightarrow[\text{translation of 2 units in the negative } x\text{-direction}]{f(x) \to f(x+2)} f(x) = (x+2)^3 \xrightarrow[\text{stretch of factor } \frac{1}{4} \parallel \text{ to } y\text{-axis}]{f(x) \to 4f(x)} f(x) = \frac{1}{4}(x+2)^3$

$\therefore$ Equation of the new curve: $y = \frac{1}{4}(x+2)^3$

(b)



3 **(a)** Sketch the curve with equation $y = x - \frac{12}{x}$, giving the exact coordinates of the point(s) where the curve crosses the axes and the equations of any asymptotes. [4]

(b) Hence, or otherwise, solve the inequality $x - \frac{12}{x} < 1$. [3]

(a) $y = \frac{x^2 - 12}{x} = \frac{(x - \sqrt{12})(x + \sqrt{12})}{x}$

Vertical asymptote: Let denominator $= 0$

i.e. $x = 0$

Oblique asymptote: Let $x \to \pm\infty$

$\therefore \frac{12}{x} \to 0$

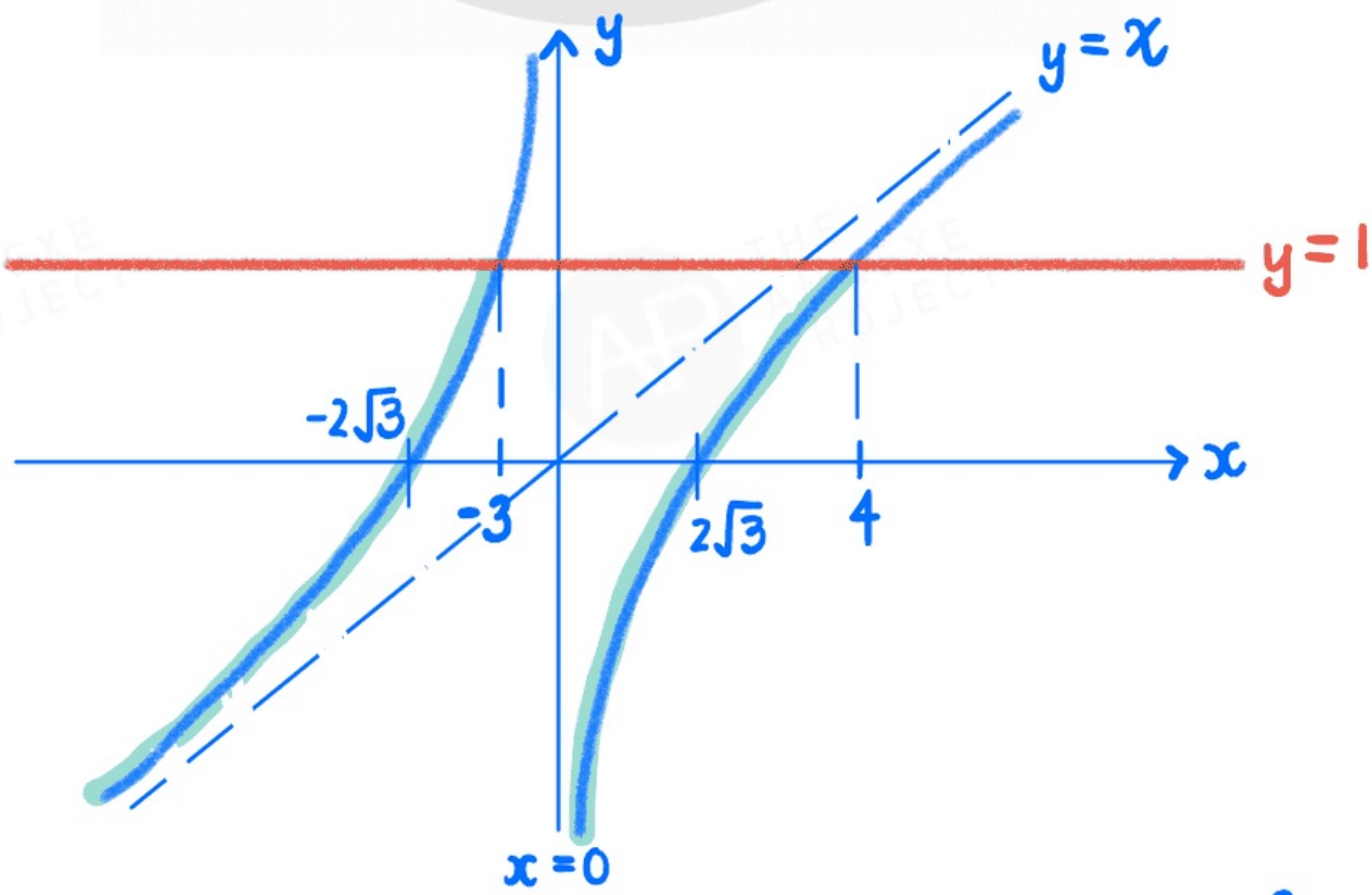
hence, $y \longrightarrow x$

i.e. $y = x$ is the oblique asymptote.

x-intercepts:

When $y = 0$, $x = \pm\sqrt{12}$

$= \pm 2\sqrt{3}$



(b) By GC, the intersection points of $y = x - \frac{12}{x}$ and $y = 1$ are at $x = -3$ and $x = 4$.

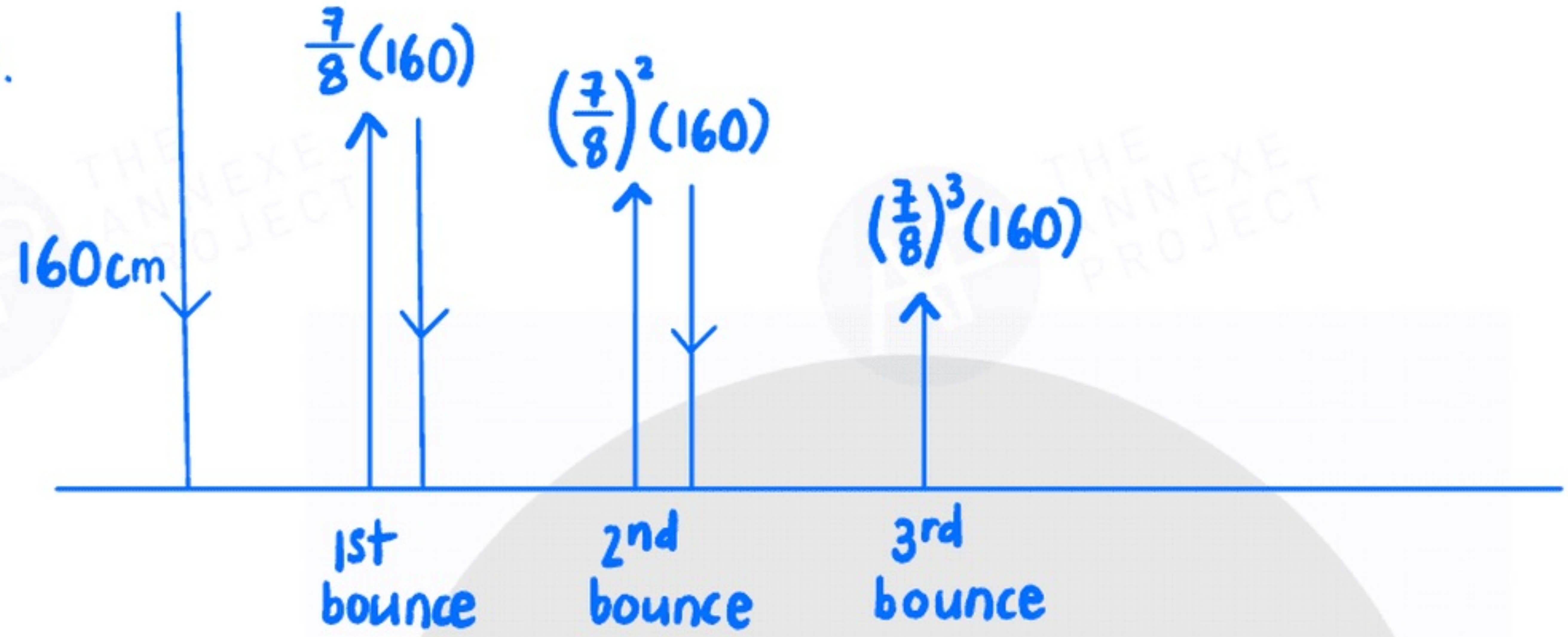
Hence, the inequality is satisfied by the highlighted part of the sketch, i.e. $x < -3$ or $0 < x < 4$

4 A student drops a ball vertically onto a hard surface and measures the height reached by the ball after each successive bounce. She drops the ball from an initial height of 160 cm and she estimates that the height the ball reaches after each bounce is $\frac{7}{8}$ of the height reached by the previous bounce.

(a) Find the total distance that the ball has travelled when it reaches the highest point after the third bounce. [2]

(b) The ball is considered to have stopped bouncing when a bounce first results in the height the ball reaches being less than 0.01 cm. Find how many bounces the ball has made and the total distance that the ball has travelled in this case. [6]

(a).



total distance when the ball reaches the highest point after the 3rd bounce:

$= 160 + 2\left(\frac{7}{8}\right)(160) + 2\left(\frac{7}{8}\right)^2(160) + \left(\frac{7}{8}\right)^3(160)$

$= 792.1875$

$= \underline{792 \text{ cm}}$

(b). Let $\left(\frac{7}{8}\right)^n (160) < 0.01$

$\left(\frac{7}{8}\right)^n < \frac{0.01}{160}$

$n > \dfrac{\lg\left(\frac{0.01}{160}\right)}{\lg\left(\frac{7}{8}\right)}$

$n > 72.5$

On the 73rd bounce, the height reached by the ball < 0.01cm.
i.e. <u>the ball made 73 bounces.</u>

total distance travelled:

$$= 160 + 2\left(\frac{7}{8}\right)(160) + 2\left(\frac{7}{8}\right)^2(160) + \cdots + 2\left(\frac{7}{8}\right)^{72}(160) + \left(\frac{7}{8}\right)^{73}(160)$$

$$= 160 + 320\left[\frac{7}{8} + \left(\frac{7}{8}\right)^2 + \cdots + \left(\frac{7}{8}\right)^{72}\right] + \left(\frac{7}{8}\right)^{73}(160)$$

$$= 160 + 320\left[\frac{\frac{7}{8}\left(1-\left(\frac{7}{8}\right)^{72}\right.}{1-\frac{7}{8}}\right] + \left(\frac{7}{8}\right)^{73}(160)$$

$$= 2399.859784$$

$$= \underline{2400 \text{ cm}}$$

5 The curve C has equation $y = \frac{1}{x}(\ln x)^3$, where $x > 1$.

(a) Find the exact x-coordinate, $x = x_1$, of the turning point on C and determine its nature. [4]

(b) Using calculus, find the exact area of the region between C, the line $y = 0$ and the lines with equations $x = \text{e}$ and $x = x_1$. [3]

(a) $y = \frac{1}{x}(\ln x)^3$

$$\frac{dy}{dx} = \frac{1}{x}(3)(\ln x)^2\left(\frac{1}{x}\right) + (\ln x)^3\left(\frac{-1}{x^2}\right)$$

$$= \frac{(\ln x)^2}{x^2}(3 - \ln x)$$

Let $\frac{dy}{dx} = 0$: $(\ln x)^2 = 0$ or $3 - \ln x = 0$

$\ln x = 0$ $\quad$ $\ln x = 3$

$x = e^0$ $\quad$ $x = e^3$

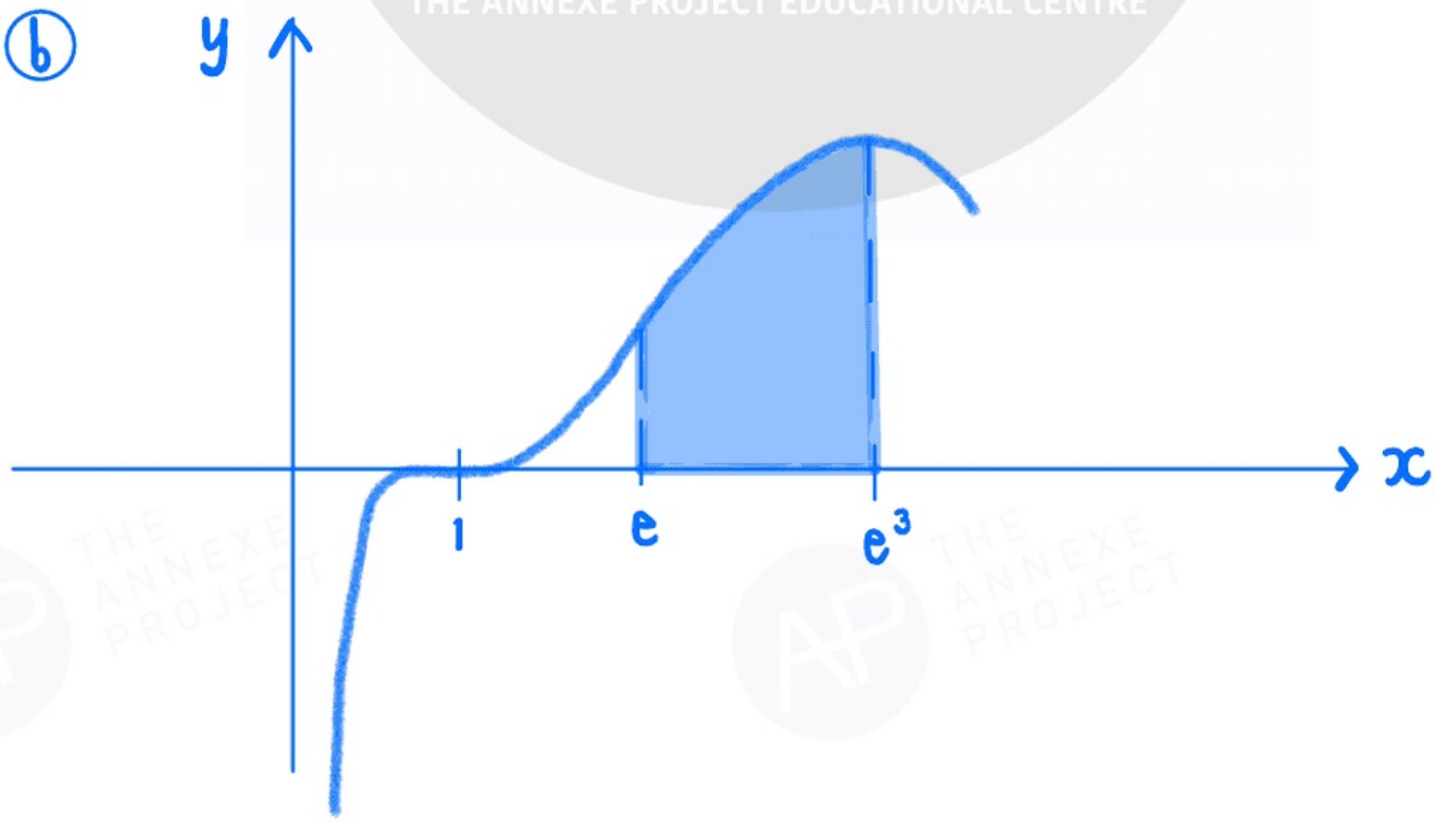
$= 1$

(Rej. because $x > 1$)

x	20.05	e^3	20.1
$\frac{dy}{dx}$	3.96×10^{-5}	0	-1.60×10^{-5}
	/	—	\

At $x = e^3$, it is a maximum point

(b)



The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

$$A = \int_{e}^{e^3} \underbrace{\frac{1}{x}}_{f'(x)} \underbrace{(\ln x)^3}_{f(x)^n} \, dx$$

$$= \left[\frac{(\ln x)^4}{4} \right]_{e}^{e^3}$$

$$= \frac{1}{4}\left[(\ln e^3)^4 - (\ln e)^4 \right]$$

$$= \frac{1}{4}\left[3^4 - 1 \right]$$

$$= \underline{20 \text{ sq. units}}$$

★ Reminder:

$$\int f'(x) f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$$

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6 **(a)** The non-zero vectors **a**, **b** and **c** are such that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$. Given that $\mathbf{b} \neq -\mathbf{c}$, find a linear relationship between **a**, **b** and **c**. [3]

(b) The variable position vector **v** satisfies the equation $\mathbf{v} \times (\mathbf{i} - 3\mathbf{k}) = 2\mathbf{j}$.

(i) Find the set of position vectors **v**. [3]

(ii) Describe the set of points represented by position vectors **v** geometrically. [2]

(a) Given $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{c} \times \underset{\sim}{a}$

$$(\underset{\sim}{a} \times \underset{\sim}{b}) - (\underset{\sim}{c} \times \underset{\sim}{a}) = 0$$

$$(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{a} \times \underset{\sim}{c}) = 0$$

$$\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = 0$$

Hence, $\underset{\sim}{a} \parallel (\underset{\sim}{b} + \underset{\sim}{c})$

$\therefore \underset{\sim}{a} = k(\underset{\sim}{b} + \underset{\sim}{c})$, $k \in \mathbb{R}$

(b) i. Let $\underset{\sim}{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -3b \\ c + 3a \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

by comparison, $b = 0$, $3a = 2 - c$

$$a = \frac{2}{3} - \frac{1}{3}c$$

we can express $\underset{\sim}{v} = \left\{ \begin{pmatrix} \frac{2}{3} - \frac{1}{3}c \\ 0 \\ c \end{pmatrix}, c \in \mathbb{R} \right\}$

ii. Let c be a parameter λ:

then $\underset{\sim}{v} = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

v is the set of position vectors that satisfy the line equation of $l: \underset{\sim}{r} = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

7 Do not use a calculator in answering this question.

(a) Showing your working, find the complex numbers z and w which satisfy the simultaneous equations.

$$2\mathrm{i}z + (1 - 2\mathrm{i})w = 4$$
$$(1 + \mathrm{i})z + (2 + \mathrm{i})w = 3$$

[6]

$$2\mathrm{i}z + (1 - 2\mathrm{i})w = 4 \quad \text{—— } (1)$$
$$(1 + \mathrm{i})z + (2 + \mathrm{i})w = 3 \quad \text{—— } (2)$$

$(1) \times i$: $2i^2 z + (i - 2i^2)w = 4i$

$-2z + iw + 2w = 4i$

$-2z + (2+i)w = 4i$

$(2+i)w = 2z + 4i$ —— (3)

Sub (3) into (2):

$(1+i)z + 2z + 4i = 3$

$(3+i)z = 3 - 4i$

$z = \dfrac{3-4i}{3+i}$

$= \dfrac{1}{2} - \dfrac{3}{2}i$

(3): $(2+i)w = 2\left(\frac{1}{2} - \frac{3}{2}i\right) + 4i$

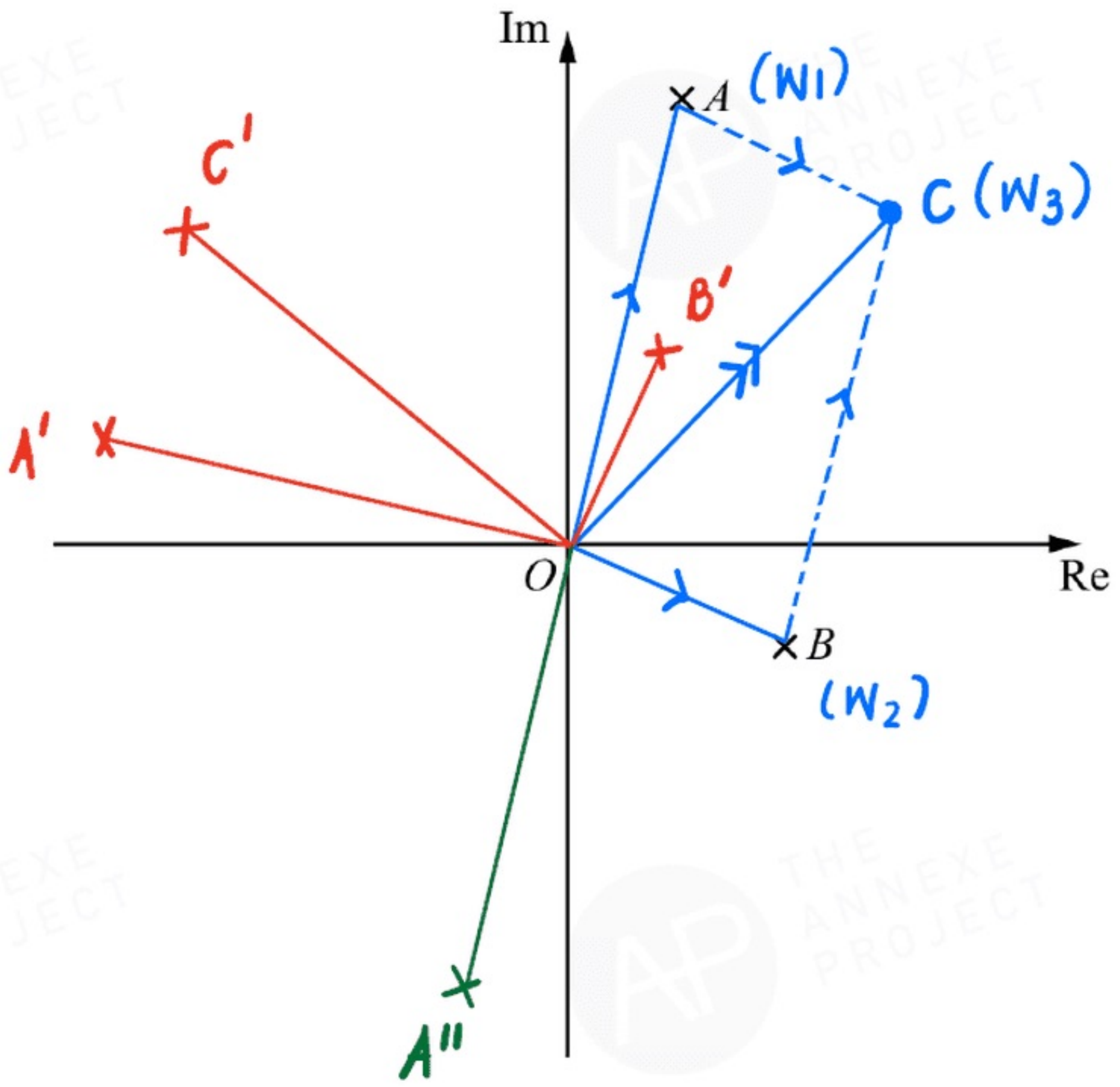
$(2+i)w = 1 - 3i + 4i$

$(2+i)w = (1+i)$

$w = \dfrac{1+i}{2+i}$

$= \dfrac{3}{5} + \dfrac{1}{5}i$

(b)



The points A and B on the Argand diagram represent the complex numbers w_1 and w_2 respectively.

(i) On the copy of the Argand diagram in the Printed Answer Booklet, plot the point C to represent the complex number w_3, where $w_3 = w_1 + w_2$. Show clearly the geometrical relationship between the points A, B and C. [2]

The points A', B' and C' represent the complex numbers $\mathrm{i}w_1$, $\mathrm{i}w_2$ and $\mathrm{i}w_3$ respectively.

(ii) On the same Argand diagram in the Printed Answer Booklet, plot the points A', B' and C'. State the transformation that maps the points A, B and C onto the points A', B' and C'. [2]

(iii) The transformation in **part (ii)** maps point A' onto the point A''. Determine, with justification, whether A'' represents the complex conjugate of w_1. [1]

(ii) $90°$ anti-clockwise rotation about O.

(iii) $A \xrightarrow[\text{about } O]{180° \text{ rotation}} A''$.

As seen from the argand diagram above, A'' is not a reflection of A about the real axis, hence $A'' \neq$ complex conjugate of w_1.

8 The curve C has parametric equations

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad \text{where } 0 < t < \tfrac{1}{2}\pi.$$

(a) Find the equation of the tangent to C at the point P with parameter p. [3]

(b) The tangent at P meets the x-axis at the point A and meets the y-axis at the point B. Find the length of AB. [3]

(c) Show that the cartesian equation of C may be written as $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. [2]

(d) The region bounded by C, the line $x = 0$ and the line $y = 0$ is rotated through 2π about the y-axis. Find the volume of revolution of the solid formed, giving your answer correct to 3 decimal places. [4]

(a) $x = (\cos t)^3$

$\frac{dx}{dt} = 3(\cos t)^2(-\sin t)$

$= -3\sin t\cos^2 t$

$y = (\sin t)^3$

$\frac{dy}{dt} = 3(\sin t)^2 \cos t$

$\therefore \frac{dy}{dx} = \frac{\cancel{3}\sin^{\cancel{2}}t\,\cancel{\cos t}}{-\cancel{3}\cancel{\sin t}\cos^{\cancel{2}}t} = \underline{-\tan t}$

When $t = p$, $x = \cos^3 p$, $y = \sin^3 p$, $\frac{dy}{dx} = -\tan p$

Eqn of tangent:

$y - \sin^3 p = (-\tan p)(x - \cos^3 p)$

$y = -(\tan p)x + \sin p\cos^2 p + \sin^3 p$

$= -(\tan p)x + \sin p(\cos^2 p + \sin^2 p)$

$= \underline{-(\tan p)x + \sin p}$

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(b) A: Let $y = 0$

$\therefore (\tan p)x = \sin p$

$x = \sin p \times \frac{\cos p}{\sin p}$

$= \cos p$

$\underline{A = (\cos p, 0)}$

B: Let $x = 0$

$\therefore y = \sin p$

$\underline{B = (0, \sin p)}$

Length $AB = \sqrt{(\cos p - 0)^2 + (0 - \sin p)^2}$

$= \sqrt{\cos^2 p + \sin^2 p} = \underline{1 \text{ unit}}$

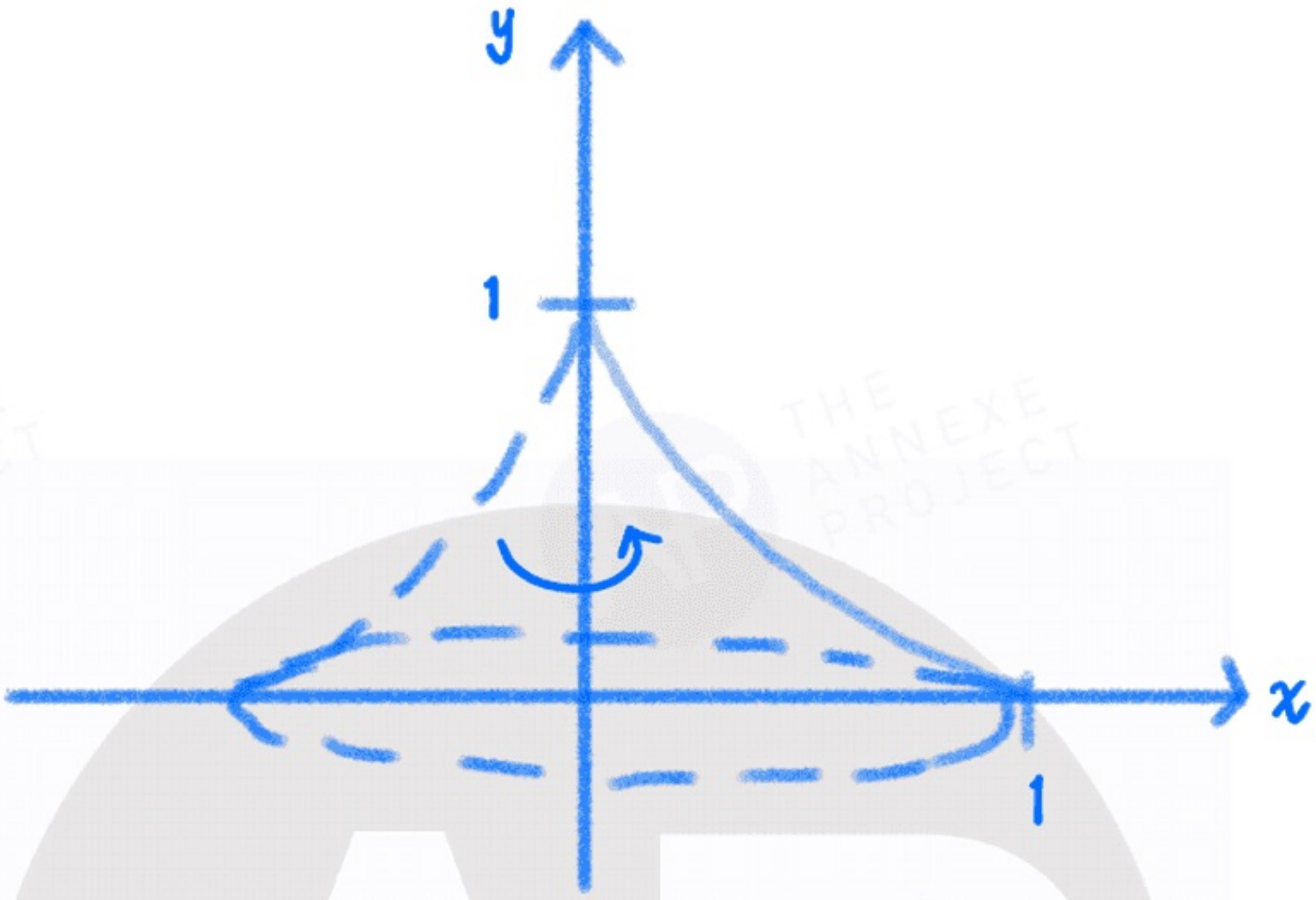
(c) $x = (\cos t)^3$ $\qquad$ $y = (\sin t)^3$

$\therefore \cos t = x^{\frac{1}{3}}$ $\qquad$ $\sin t = y^{\frac{1}{3}}$

$\boxed{\text{Using identity: } \sin^2 t + \cos^2 t = 1}$

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = 1 \text{ (shown)}$$

(d)

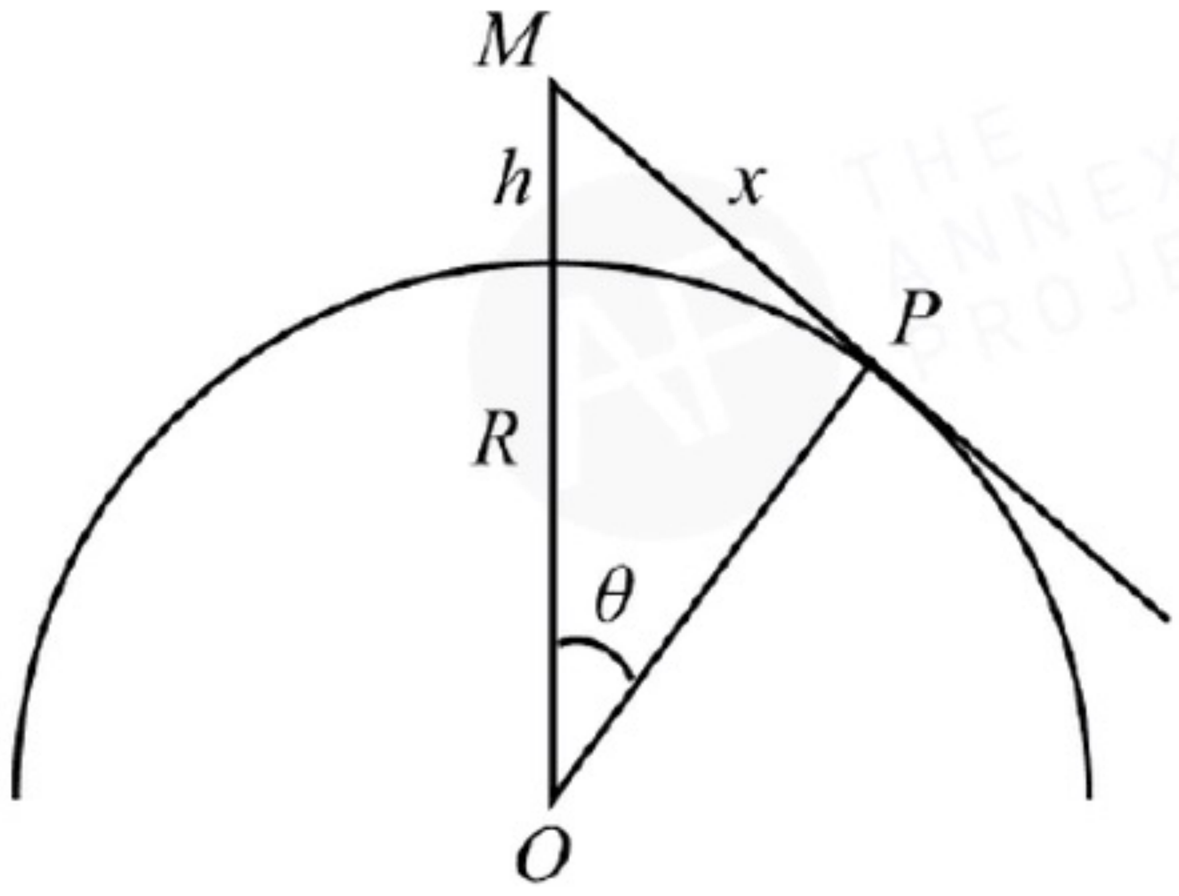


$$V = \pi \int_{y=0}^{y=1} x^2 \, dy \qquad x^{\frac{2}{3}} = 1 - y^{\frac{2}{3}}$$

$$= \pi \int_{y=0}^{y=1} (1 - y^{\frac{2}{3}})^3 \, dy \qquad \therefore x^2 = (1 - y^{\frac{2}{3}})^3$$

$= \underline{0.479 \text{ cubic units.}}$ (3 d.p.)

9



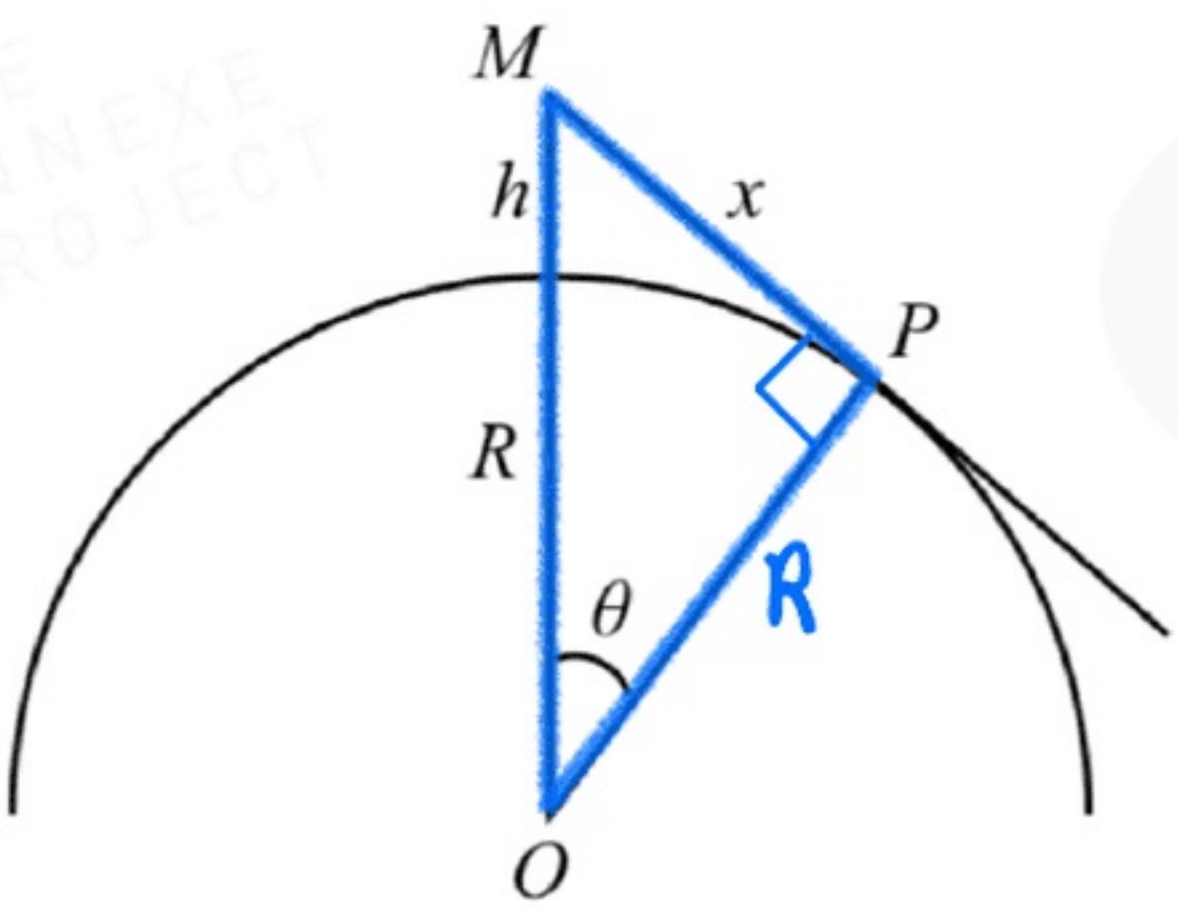
A man M is at the top of a mountain which is of height h km. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the man can see is a point P such that $MP = x$ km and the angle $POM = \theta$, where O is the centre of the earth (see diagram). You may assume that the height of the man is negligible.

(a) Show that $x = (2hR)^{\frac{1}{2}}\left(1 + \dfrac{h}{2R}\right)^{\frac{1}{2}}$. [3]

(b) It is given that h is small compared to R. If $\alpha = \dfrac{h}{R}$, show that $\sin\theta \approx (2\alpha)^{\frac{1}{2}}\left(1 - \dfrac{3}{4}\alpha\right)$. [5]

(c) The man M has a scientific instrument which enables him to estimate the angle between PM and the horizontal. Given that this angle is $2°$ and that the radius of the earth is 6375 km, find estimates for the values of α and h. [4]

(a)



Using pythagoras' theorem

$$(h+R)^2 = x^2 + R^2$$

$$h^2 + 2hR + \cancel{R^2} = x^2 + \cancel{R^2}$$

$$x^2 = h^2 + 2hR$$

$$= (2hR)\left[1 + \frac{h^2}{2hR}\right]$$

$$= (2hR)\left(1 + \frac{h}{2R}\right)$$

$$\therefore \quad x = \sqrt{(2hR)\left(1 + \frac{h}{2R}\right)}$$

$$= (2hR)^{\frac{1}{2}}\left(1 + \frac{h}{2R}\right)^{\frac{1}{2}}$$

(b)

$$\sin\theta = \frac{x}{h+R}$$

$$= \frac{(2hR)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{h+R}$$

divide both numerator and denominator by R.

$$= \frac{\frac{(2hR)^{\frac{1}{2}}}{R}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{\left(\frac{h}{R}+1\right)}$$

$$= \frac{\left(\frac{2hR}{R^2}\right)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{\left(\frac{h}{R}+1\right)}$$

$$= \frac{\left(\frac{2h}{R}\right)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{\left(\frac{h}{R}+1\right)}$$

$$= \frac{(2\alpha)^{\frac{1}{2}}\left(1+\frac{1}{2}\alpha\right)^{\frac{1}{2}}}{(1+\alpha)}$$

$$= (2\alpha)^{\frac{1}{2}}\left(1+\frac{1}{2}\alpha\right)^{\frac{1}{2}}(1+\alpha)^{-1}$$

$$= (2\alpha)^{\frac{1}{2}}\left[1+\frac{1}{2}\left(\frac{1}{2}\alpha\right)+\cdots\right]\left[1-\alpha+\cdots\right]$$

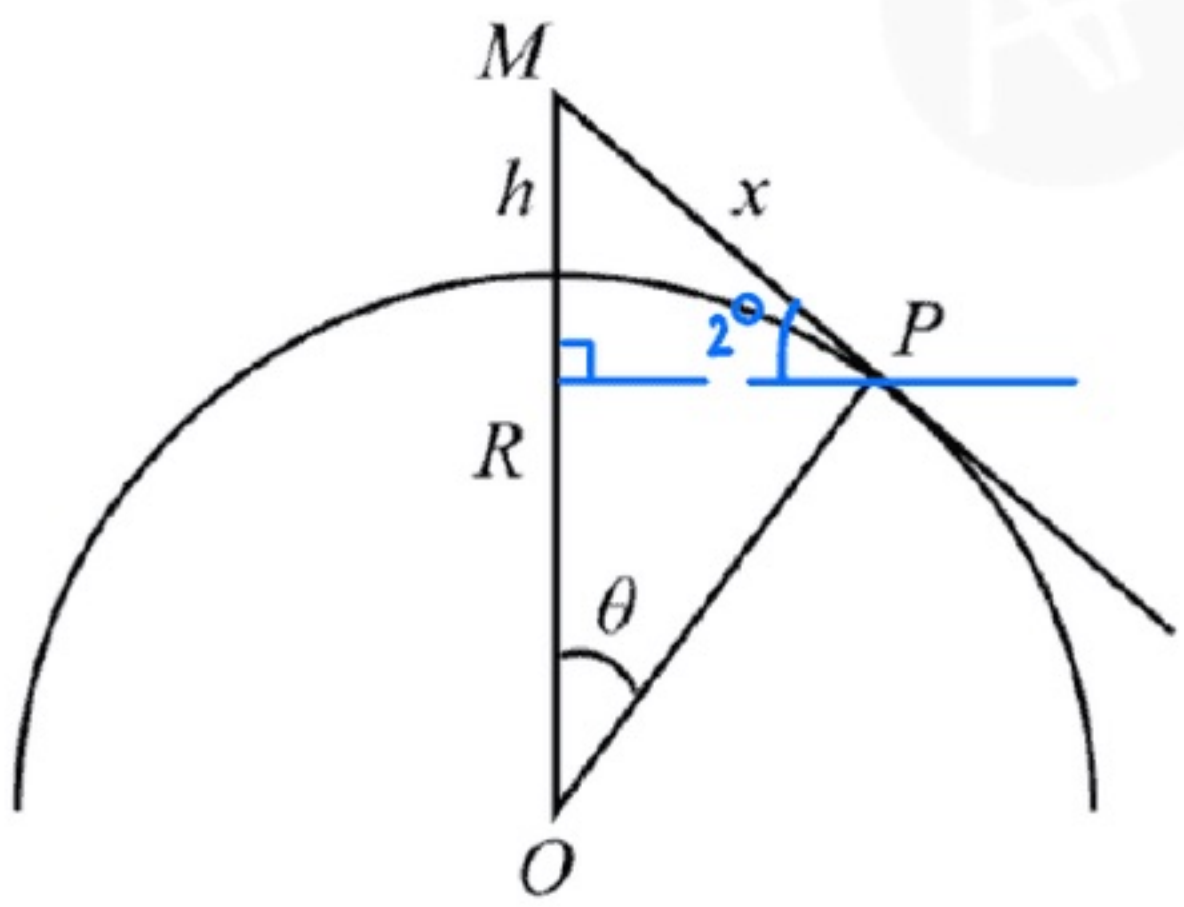
$$\approx (2\alpha)^{\frac{1}{2}}\left(1+\frac{1}{4}\alpha\right)(1-\alpha)$$

If h is small, α is small, hence α^2 and above are negligible.

$$= (2\alpha)^{\frac{1}{2}}\left(1-\alpha+\frac{1}{4}\alpha+\cdots\right)$$

$$\approx (2\alpha)^{\frac{1}{2}}\left(1-\frac{3}{4}\alpha\right)$$

(c)



By similar $\triangle$s:

$\theta = 2^\circ$

i.e. $\sin 2^\circ \approx (2\alpha)^{\frac{1}{2}}\left(1-\frac{3}{4}\alpha\right)$

By GC: $\alpha = 6.0954\times10^{-4}$

or $\alpha = 1.3045$ (rej. because α is small).

$$\alpha = \frac{h}{R}$$

$\therefore h = \alpha R = 6.0954\times10^{-4}\times 6375$

$= 3.89$ km

10 The point A has coordinates $(0, 2, -3)$. The line l has equation $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$.

(a) Find the cartesian equation of the plane π which contains A and is perpendicular to l. [3]

(b) Hence, or otherwise, find the coordinates of the point P on l which is closest to A. [3]

(c) The line m passes through the point with coordinates $(4, -5, 10)$ and P. The line n lies in the same plane as l and m. Find a cartesian equation for n if n is the reflection of the line m the line l. [6]

$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $l : \underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

(a)

$\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$= -6 - 3$

$= -9$

$\therefore \underline{2x - 3y + z = -9}$

(b) Sub eqn. of l into eqn. of plane to find P:

$\begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9$

$4\lambda + 3 + 9\lambda + 2 + \lambda = -9$

$14\lambda = -14$

$\lambda = -1$

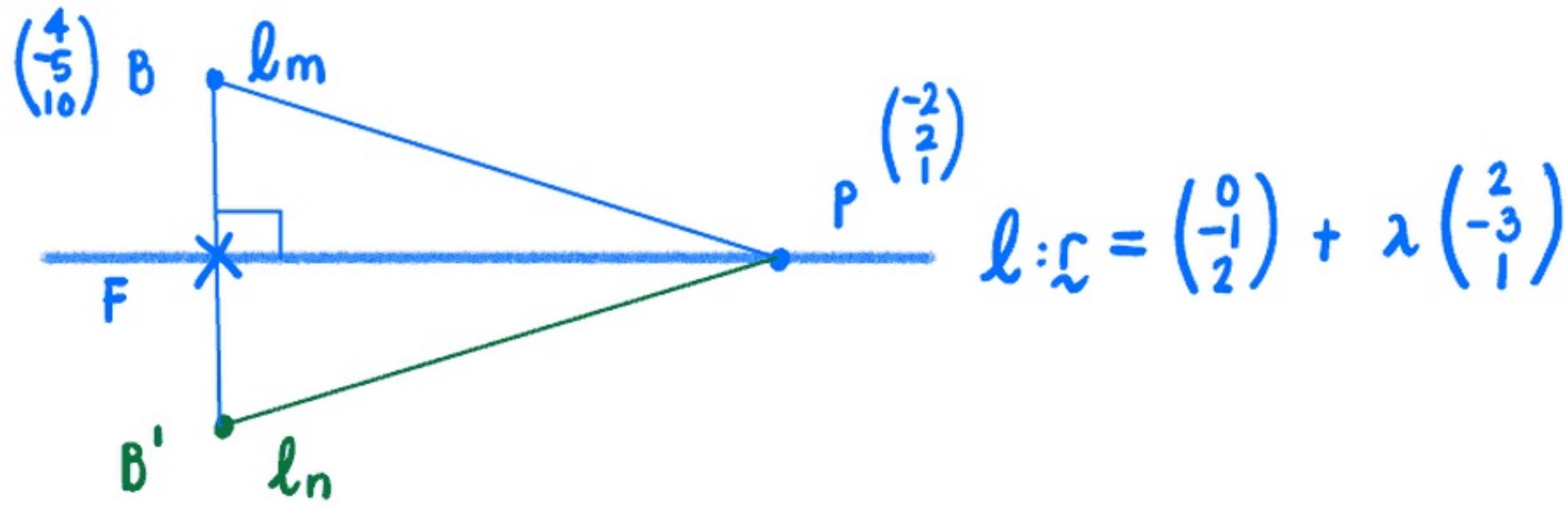
$\therefore$ we sub $\lambda = -1$ into eqn. of l, i.e. $\begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix}$

$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $\underline{\text{coordinates of } P = (-2, 2, 1)}$

(c) Step 1:

Let $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix}$

then $\overrightarrow{BP} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ -9 \end{pmatrix}$



Step 2: Find foot of B on l, i.e. F

Let $\overrightarrow{OF} = \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix}$

then $\overrightarrow{BF} = \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} -4+2\lambda \\ 4-3\lambda \\ -8+\lambda \end{pmatrix}$

$\overrightarrow{BF} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$ (since $\overrightarrow{BF} \perp l$)

$\begin{pmatrix} -4+2\lambda \\ 4-3\lambda \\ -8+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \Rightarrow -8+4\lambda-12+9\lambda-8+\lambda = 0$

$14\lambda = 28$

$\lambda = 2$

$\therefore \overrightarrow{OF} = \begin{pmatrix} 4 \\ -7 \\ 4 \end{pmatrix}$

Step 3: Using Ratio Theorem to find B'

$\overrightarrow{OF} = \dfrac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$

$\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB} = 2\begin{pmatrix} 4 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix}$

Step 4: direction vector of l_n

$\overrightarrow{B'P} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 11 \\ 3 \end{pmatrix}$

$l_n: \underset{\sim}{r} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 11 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}.$

$x = -2-6\mu, \quad y = 2+11\mu, \quad z = 1+3\mu$

$\dfrac{-x-2}{6} = \dfrac{y-2}{11} = \dfrac{z-1}{3}$ (cartesian eqn. of l_n).

11 The variable A is such that the rate of increase of A with respect to time t is proportional to the product of A and $(10 - A)$. The initial value of A is 2 and when $t = 5$ the value of A is 4.

(a) Write down a differential equation expressing the relation between A and t. Find the time at which $A = 8$, giving your answer correct to 2 decimal places. [8]

(b) Find the value of A when $t = 24$, giving your answer correct to 2 decimal places. [2]

(c) Write the solution of the differential equation in the form $A = \mathrm{f}(t)$ and sketch the graph of A against t. [4]

(a)

$$\frac{dA}{dt} = kA(10-A)$$

$$\int \frac{1}{A(10-A)}\,dA = k\int dt$$

$$\frac{1}{10}\int \frac{1}{A} + \frac{1}{10-A}\,dA = kt + C$$

$$\ln|A| - \ln|10-A| = 10kt + 10C$$

$$\ln\left|\frac{A}{10-A}\right| = 10kt + 10C$$

$$\frac{A}{10-A} = \pm e^{10kt+10C}$$

$$= \pm e^{10C} e^{10kt}$$

$$= Be^{10kt} \quad (\text{where } B = \pm e^{10C})$$

$$A = (10-A)Be^{10kt}$$

$$A + ABe^{10kt} = 10Be^{10kt}$$

$$A = \frac{10Be^{10kt}}{(1+Be^{10kt})}$$

By Partial Fractions

$$\frac{1}{A(10-A)} = \frac{a}{A} + \frac{b}{10-A}$$

By Cover-up rule:

$a = \frac{1}{10}$, $b = \frac{1}{10}$

When $t = 0$, $A = 2$:

$$2 = \frac{10B}{1+B}$$

$$2 + 2B = 10B$$

$$8B = 2$$

$$B = \frac{1}{4}$$

When $t = 5$, $A = 4$:

$$4 = \frac{10(\frac{1}{4})e^{50k}}{1+\frac{1}{4}e^{50k}}$$

$$4 + e^{50k} = 2.5e^{50k}$$

$$1.5e^{50k} = 4$$

$$e^{50k} = \frac{8}{3}$$

$$50k = \ln\frac{8}{3}$$

$$k = \frac{1}{50}\ln\frac{8}{3} \text{ or } 0.019617$$

Hence, $A = \dfrac{2.5e^{0.19617t}}{1+0.25e^{0.19617t}}$

<u>When $A = 8$</u> :

$$8 + 2e^{0.19617t} = 2.5e^{0.19617t}$$
$$0.5e^{0.19617t} = 8$$
$$e^{0.19617t} = 16$$
$$0.19617t = \ln 16$$
$$\therefore \underline{t = 14.13}$$

ⓑ <u>When $t = 24$</u> :

$$A = \frac{2.5e^{0.19617(24)}}{1+0.25e^{0.19617(24)}} = \underline{9.65}$$

ⓒ $$A = \frac{2.5e^{0.19617t}}{1+0.25e^{0.19617t}} = \frac{2.5}{\frac{1}{e^{0.19617t}} + 0.25}$$
$$= \frac{10}{4e^{-0.19617t}+1}$$

As $t \to -\infty$, $e^{-0.19617t} \to \infty$, $A \to 0$

As $t \to \infty$, $e^{-0.19617t} \to 0$, $A \to 10$