

MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE
NAME

--

CENTRE
NUMBER

S				
---	--	--	--	--

INDEX
NUMBER

--	--	--	--

MATHEMATICS

9758/02

Paper 2

October/November 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.

Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

DO NOT WRITE ON ANY BARCODES.

Answer **all** the questions.

Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

- 1 Use the substitution $u = \sqrt{x+2}$ to find $\int \frac{x}{\sqrt{x+2}} dx$.

[4]

Step 1 : $u = (x+2)^{\frac{1}{2}}$
 $\frac{du}{dx} = \frac{1}{2}(x+2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+2}}$

Step 2 : $\int \frac{x}{\sqrt{x+2}} dx$
 $= \int \frac{u^2 - 2}{u} \cdot 2u du$
 $= 2 \int u^2 - 2 du$
 $= \frac{2u^3}{3} - 4u + C$
 $= \frac{2}{3}(x+2)^{\frac{3}{2}} - 4\sqrt{x+2} + C$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+2}}$$

$$\frac{du}{dx} = \frac{1}{2u}$$

$$\int dx = \int 2u du$$



THE ANNEXE PROJECT
EDUCATIONAL CENTRE

- 2 A cylindrical container has a base of area 0.9 m^2 and height $h \text{ m}$. Water is poured into the container at the rate of k litres per second. The container is filled in 72 seconds.

$$[1 \text{ m}^3 = 1000 \text{ litres}]$$

- (a) Show that $k = 12.5h$.

[1]

The container is emptied. Water is now poured into the container at the rate of kt litres per second, where t is the time in seconds from when pouring begins.

- (b) Find the time taken to fill the container.

[3]

① Vol. of container = $0.9h \text{ m}^3$

Hence, $0.9h = \frac{k \times 72}{1000}$

$$900h = 72k$$

$$k = \frac{900h}{72}$$

$$= 12.5h \text{ (shown)}$$

- ② Let V (in litres) be the vol. of water in the container at any time t (in seconds)

$$\frac{dV}{dt} = kt$$

$$V = \int kt \, dt$$

$$V = \frac{kt^2}{2} + C$$

When $t = 0$, $V = 0 \text{ l}$: $C = 0$

Hence, $V = \frac{1}{2}kt^2$ where $k = 12.5h$

When $V = 0.9h \text{ m}^3$, i.e. $V = 0.9h \times 1000 \text{ l}$
 $= 900h \text{ l}$

$$900h = \frac{1}{2}(12.5h)t^2$$

$$t^2 = 144$$

$$t = 12 \text{ s}$$

The container is emptied again. Water is now poured into the container at the rate of $(kt+25)$ litres per second, where t is the time in seconds from when pouring begins. The container is filled in 10 seconds.

(c) Find the height of the container, h .

[3]

$$\frac{dV}{dt} = kt + 25$$

$$V = \int kt + 25 \, dt$$

$$= \frac{kt^2}{2} + 25t + C$$

$$\text{When } t = 0 \text{ s}, V = 0 \text{ l}, C = 0$$

$$\text{Hence, } V = \frac{1}{2}kt^2 + 25t, \text{ where } k = 12.5h$$

$$\text{When } V = 900h \text{ l}, t = 10 \text{ s:}$$

$$900h = \frac{1}{2}(12.5h)(10)^2 + 25(10)$$

$$900h = 625h + 250$$

$$275h = 250$$

$$h = \frac{10}{11} \text{ m.}$$

THE ANNEXE PROJECT EDUCATIONAL CENTRE



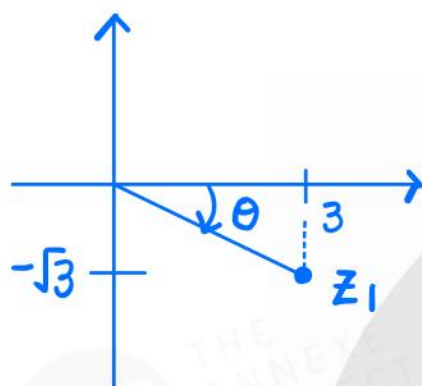
THE ANNEXE PROJECT
EDUCATIONAL CENTRE

3 The complex numbers z_1 , z_2 and z_3 are such that $z_1 = 3 - i\sqrt{3}$, $z_2 = \frac{1}{2}e^{i\frac{2\pi}{5}}$ and $z_3 = z_1 \times z_2$.

(a) Find exactly the modulus and argument of z_3 . [3]

(b) Sketch an Argand diagram showing z_1 , z_2 and z_3 . [2]

(a) $z_1 = 3 - i\sqrt{3}$. $|z_1| = \sqrt{3^2 + \sqrt{3}^2}$
 $= \sqrt{9 + 3}$
 $= 2\sqrt{3}$



$$\tan \theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

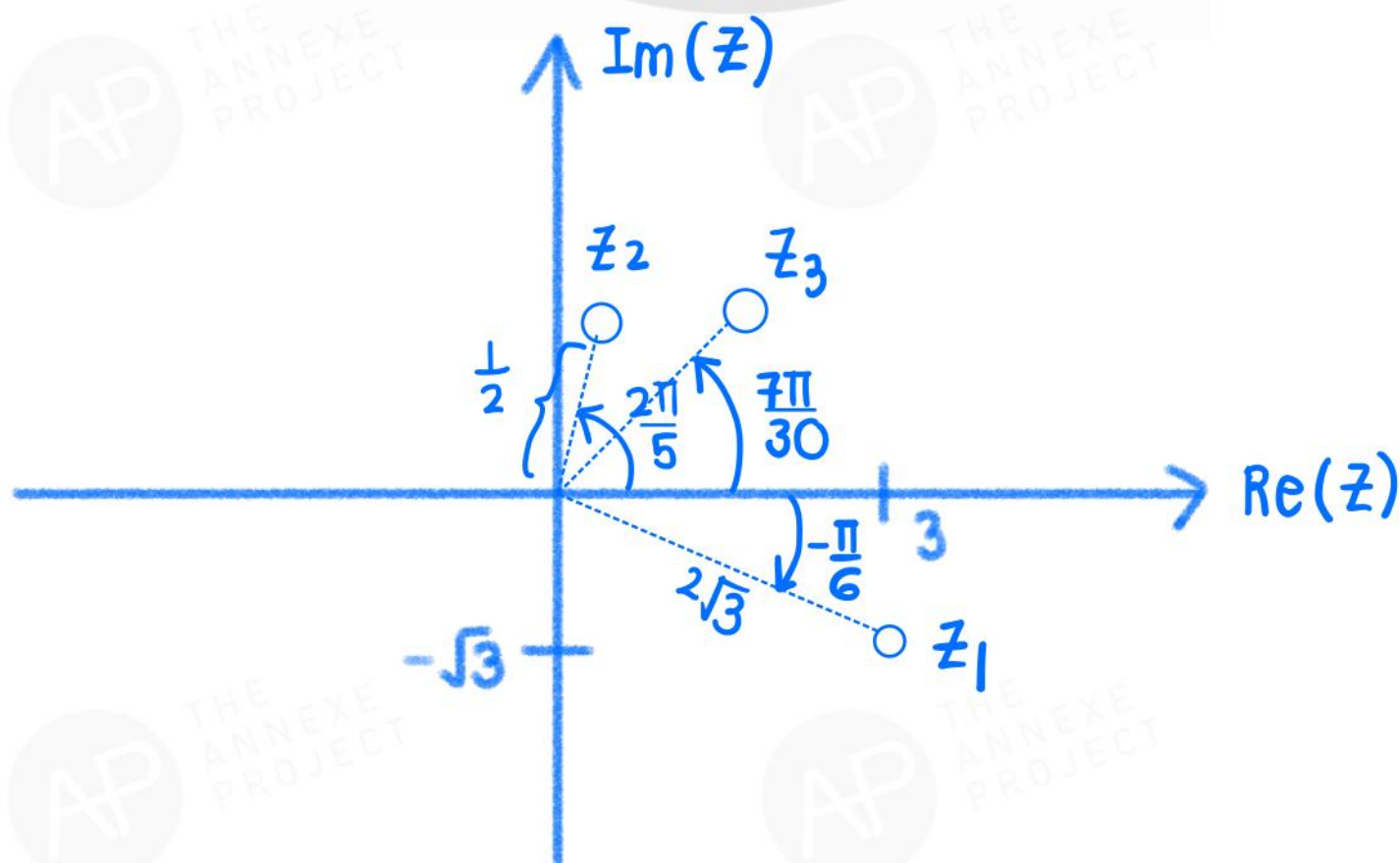
$$\therefore \arg z_1 = -\frac{\pi}{6}$$

$$z_1 = 2\sqrt{3} e^{i(-\frac{\pi}{6})}$$

$$\begin{aligned} z_3 &= z_1 \times z_2 \\ &= 2\sqrt{3} e^{i(-\frac{\pi}{6})} \cdot \frac{1}{2} e^{i(\frac{2\pi}{5})} \\ &= \sqrt{3} e^{i(-\frac{\pi}{6} + \frac{2\pi}{5})} \\ &= \sqrt{3} e^{i(\frac{7\pi}{30})} \end{aligned}$$

$$|z_3| = \sqrt{3} , \arg z_3 = \frac{7\pi}{30}$$

(b)



- (c) Find the smallest positive integer value of n for which z_3^n is purely imaginary. State the modulus and argument of z_3^n in this case, giving the modulus in the form $k\sqrt{3}$, where k is an integer. [4]

$$\begin{aligned} z_3^n &= \sqrt{3}^n e^{i \frac{7n\pi}{30}} \\ &= \sqrt{3}^n \left[\cos \frac{7n\pi}{30} + i \sin \frac{7n\pi}{30} \right] \end{aligned}$$

If z_3^n is purely imaginary,

$$\cos \frac{7n\pi}{30} = 0,$$

$$\text{Hence, } \frac{7n\pi}{30} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$= \frac{\pi}{2} (2k+1), \quad k \in \mathbb{Z}_0^+$$

$$\frac{7n}{30} = \frac{2k+1}{2}$$

$$n = \frac{15(2k+1)}{7}, \quad k \in \mathbb{Z}_0^+$$

Since n is a positive integer value,

$$n = 15 \quad \text{when } k = 3$$

When $n = 15$,

$$|z_3^n| = \sqrt{3}^{15} = \sqrt{3}^{14} \cdot \sqrt{3} = 3^7 \sqrt{3} = 2187\sqrt{3}$$

$$\arg z_3^n = n \arg z_3 = 15 \times \frac{7\pi}{30} = \frac{7\pi}{2}$$

(out of principal range)

$$\text{Hence, } \arg z_3^n = \frac{7\pi}{2} - 4\pi = -\frac{\pi}{2}$$

4 (a) Write $\frac{1}{9r^2+3r-2}$ in partial fractions.

[2]

(b) Find an expression in terms of m for $\sum_{r=m}^{3m} \left(\frac{1}{9r^2+3r-2} \right)$. Write your answer as a single fraction in terms of m .

[4]

$$\begin{aligned} \textcircled{a} \quad \frac{1}{(3r-1)(3r+2)} &= \frac{A}{(3r-1)} + \frac{B}{(3r+2)} \\ &= \frac{A(3r+2) + B(3r-1)}{(3r-1)(3r+2)} \end{aligned}$$

comparing numerators

$$1 = A(3r+2) + B(3r-1)$$

$$\begin{aligned} \text{sub } r = -\frac{2}{3} : \quad 1 &= B(-2-1) \\ B &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{sub } r = \frac{1}{3} : \quad 1 &= A(1+2) \\ A &= \frac{1}{3} \end{aligned}$$

$$\frac{1}{(3r-1)(3r+2)} = \frac{1}{3(3r-1)} - \frac{1}{3(3r+2)}$$

$$\begin{aligned} \textcircled{b} \quad \sum_{r=m}^{3m} \left(\frac{1}{9r^2+3r-2} \right) &= \sum_{r=m}^{3m} \frac{1}{3(3r-1)} - \frac{1}{3(3r+2)} \\ &= \frac{1}{3} \sum_{r=m}^{3m} \frac{1}{3r-1} - \frac{1}{3r+2} \\ &= \frac{1}{3} \left[\left(\frac{1}{3m-1} - \frac{1}{3m+2} \right) \right. \\ &\quad \left. + \left(\frac{1}{3m+2} - \frac{1}{3m+5} \right) \right. \\ &\quad \left. + \left(\frac{1}{3m+5} - \frac{1}{3m+8} \right) \right. \\ &\quad \left. + \dots \right. \\ &\quad \left. + \left(\frac{1}{9m-1} - \frac{1}{9m+2} \right) \right] \\ &= \frac{1}{3} \left[\frac{1}{3m-1} - \frac{1}{9m+2} \right] \\ &= \frac{1}{3} \left[\frac{9m+2 - (3m-1)}{(3m-1)(9m+2)} \right] \\ &= \frac{1}{3} \left[\frac{6m+3}{(3m-1)(9m+2)} \right] \\ &= \frac{2m+1}{(3m-1)(9m+2)} \end{aligned}$$

(c) Find $\sum_{r=1}^{\infty} \left(\frac{1}{9r^2 + 3r - 2} \right)$.

[1]

(d) Find the smallest value of n for which $\sum_{r=1}^n \left(\frac{1}{9r^2 + 3r - 2} \right)$ differs from $\sum_{r=1}^{\infty} \left(\frac{1}{9r^2 + 3r - 2} \right)$ by less than 0.004.

[2]

$$\begin{aligned}
 \textcircled{c} \quad \sum_{r=1}^{\infty} \left(\frac{1}{9r^2 + 3r - 2} \right) &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{9r^2 + 3r - 2} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{3} \left[\frac{1}{3(1)-1} - \frac{1}{3n+2} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{1}{2} - \frac{1}{3n+2} \right) \\
 &= \frac{1}{3} \left(\frac{1}{2} \right) \\
 &= \frac{1}{6}
 \end{aligned}$$

$As \ n \rightarrow \infty, \ \frac{1}{3n+2} \rightarrow 0$

$$\textcircled{d} \quad \left| \sum_{r=1}^n \left(\frac{1}{9r^2 + 3r - 2} \right) - \sum_{r=1}^{\infty} \left(\frac{1}{9r^2 + 3r - 2} \right) \right| < 0.004$$

$$\left| \frac{1}{3} \left(\frac{1}{2} - \frac{1}{3n+2} \right) - \frac{1}{6} \right| < 0.004$$

$$\left| -\frac{1}{3} \left(\frac{1}{3n+2} \right) \right| < 0.004$$

$$\frac{1}{3} \left| \frac{1}{3n+2} \right| < 0.004$$

$$\left| \frac{1}{3n+2} \right| < 0.012$$

Since n is a positive integer,

$$\frac{1}{3n+2} < 0.012$$

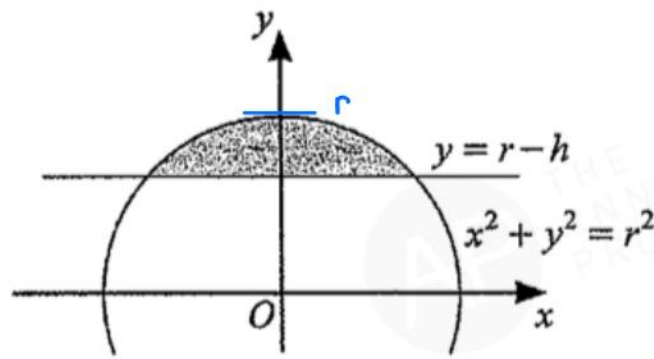
$$3n+2 > \frac{1}{0.012}$$

$$3n+2 > 83\frac{1}{3}$$

$$3n > 81\frac{1}{3}$$

$$n > 27\frac{1}{9}$$

$$\text{Least } n = 28$$



The diagram shows part of the circle $x^2 + y^2 = r^2$ and the line $y = r - h$, where $0 < h < r$. The shaded region between the circle and the line is rotated about the y -axis to form a solid, which is called a spherical cap. The height of the spherical cap is h .

(a) Show by integration that the volume of the spherical cap is $\frac{1}{3}\pi h^2(3r - h)$. [5]

Given $x^2 + y^2 = r^2$
 $x^2 = r^2 - y^2$

$$V = \pi \int_{r-h}^r (r^2 - y^2) dy$$

$$= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2(r-h) - \frac{(r-h)^3}{3} \right) \right]$$

$$= \pi \left[\frac{2}{3}r^3 - r^3 + r^2h + \frac{1}{3}(r-h)^3 \right]$$

$$= \pi \left[-\frac{1}{3}r^3 + r^2h + \frac{1}{3}(r-h)^3 \right]$$

$$= \frac{\pi}{3} \left[-r^3 + 3r^2h + (r-h)^3 \right]$$

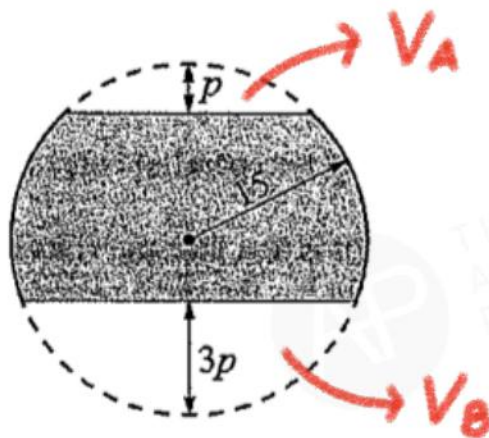
$$= \frac{\pi}{3} \left[\cancel{-r^3} + \cancel{3r^2h} + \cancel{r^3} - \cancel{3r^2h} + 3rh^2 - h^3 \right]$$

$$= \frac{\pi}{3} (3rh^2 - h^3)$$

$$= \frac{\pi}{3} h^2 (3r - h) \quad (\text{shown}).$$

$$\begin{aligned} & (r-h)^3 \\ &= (r-h)(r-h)(r-h) \\ &= (r^2 - 2rh + h^2)(r-h) \\ &= r^3 - r^2h - 2r^2h + 2rh^2 + rh^2 - h^3 \\ &= r^3 - 3r^2h + 3rh^2 - h^3 \end{aligned}$$





An ornament is made from a solid sphere of radius 15 cm by removing two spherical caps, one of height p cm from the top of the sphere and the other of height $3p$ cm from the bottom of the sphere. The plane faces of the ornament are parallel (see diagram). The volume of the ornament, shown shaded, is $3402\pi \text{ cm}^3$.

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

(b) Find the cubic equation satisfied by p , and hence find the value of p .

[4]

$$V_A = \frac{\pi}{3} p^2 (3 \times 15 - p) = \frac{\pi}{3} p^2 (45 - p)$$

$$V_B = \frac{\pi}{3} (3p)^2 (3 \times 15 - 3p) = 3\pi p^2 (45 - 3p)$$

$$\text{Vol. of ornament} = \frac{4}{3}\pi(15)^3 - V_A - V_B$$

$$3402\pi = 4500\pi - \frac{\pi}{3}p^2(45-p) - 3\pi p^2(45-3p)$$

$$3402 = 4500 - 15p^2 + \frac{1}{3}p^3 - 135p^2 + 9p^3$$

$$\therefore \frac{28}{3}p^3 - 150p^2 + 1098 = 0$$

$$28p^3 - 450p^2 + 3294 = 0$$

$$\text{By GC: } p = 15.6 \text{ (rej.) or } 3 \text{ or } -2.52 \text{ (rej.)}$$



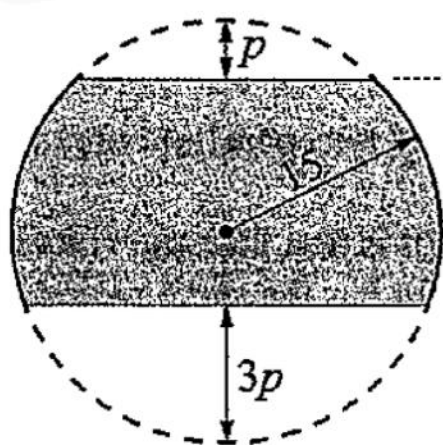
THE ANNEXE PROJECT
EDUCATIONAL CENTRE

A different ornament is made by making two parallel cuts to another sphere of radius 15 cm.

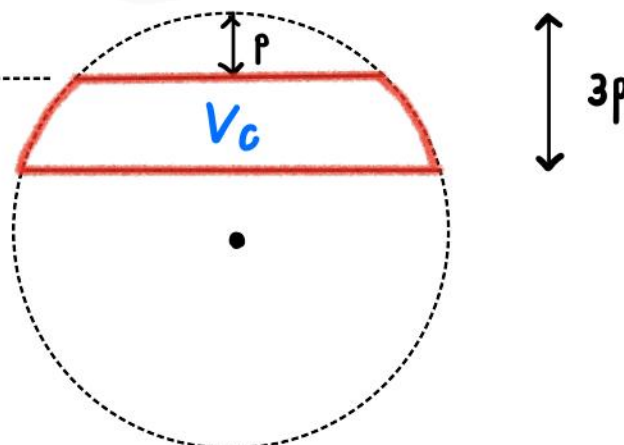
- The volume of this second ornament is less than the volume of the ornament in part (b).
- The top face of this second ornament has the same radius as the top face of the ornament in part (b).
- The bottom face of this second ornament has the same radius as the bottom face of the ornament in part (b).

(c) Find the volume of this second ornament. Give your answer as an exact multiple of π .

[2]



ornament in
part (b).



$$\begin{aligned}
 V_c &= V_B - V_A \quad (\text{from part b}). \\
 &= 3\pi p^2(45 - 3p) - \frac{\pi}{3}p^2(45 - p) \\
 &= \frac{\pi}{3}p^2[9(45 - 3p) - (45 - p)] \\
 &= \frac{\pi}{3}p^2(405 - 27p - 45 + p) \\
 &= \frac{\pi}{3}p^2(360 - 26p)
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } p = 3, \quad V_c &= \frac{\pi}{3}(3^2)(360 - 26 \times 3) \\
 &= 846\pi \text{ cm}^3
 \end{aligned}$$



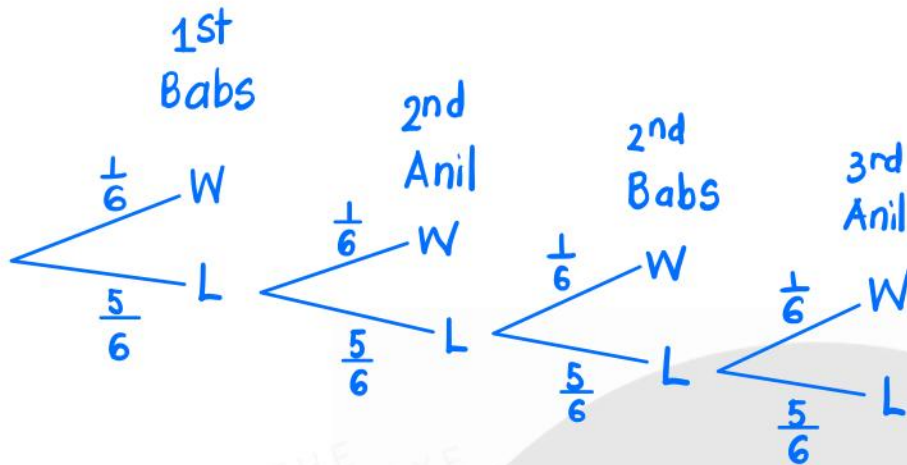
THE ANNEXE PROJECT
EDUCATIONAL CENTRE

Section B: Probability and Statistics [60 marks]

- 6 Anil and Babs are playing a game in which they take turns to throw a fair six-sided die. The winner is the first player to throw the same number as the other player has thrown immediately before. Anil throws first.

(a) Find exactly the probability that Anil wins the game. [3]

(b) Given that Babs wins the game, find exactly the probability that she wins on her second throw. [3]



$$\begin{aligned}
 \textcircled{a} \quad & \frac{5}{6} \left(\frac{1}{6} \right) + \left(\frac{5}{6} \right)^3 \left(\frac{1}{6} \right) + \left(\frac{5}{6} \right)^5 \left(\frac{1}{6} \right) + \dots \\
 &= \frac{1}{6} \left(\frac{5}{6} \right) \left[1 + \left(\frac{5}{6} \right)^2 + \left(\frac{5}{6} \right)^4 + \dots \right] \\
 &= \frac{5}{36} \left(\frac{1}{1 - \left(\frac{5}{6} \right)^2} \right) \\
 &= \frac{5}{36} \left(\frac{1}{1 - \frac{25}{36}} \right) \\
 &= \frac{5}{36} \left(\frac{1}{\frac{11}{36}} \right) \\
 &= \frac{5}{36} \cdot \frac{36}{11} \\
 &= \frac{5}{11}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{b} \quad & P(\text{Babs wins 2nd} \mid \text{Babs wins}) \\
 &= \frac{\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}}{1 - \frac{5}{11}} \\
 &= \frac{275}{1296}
 \end{aligned}$$

7 A large company is organised into 4 departments: Administration, Production, Marketing and Staffing.

- (a) The company owner wishes to find out about the qualifications of the employees in Staffing, so she gives a questionnaire to all 75 employees in this department. She receives replies from 53 of these employees. Explain whether these 53 employees form a population or a sample. [1]
- (b) The company owner also wishes to investigate the views of employees about the refreshment facilities but does not have time to ask every employee. Explain how she should carry out her investigation, and why she should do the investigation in this way. [2]

(a) A sample. Because the 53 employees is a subset (part of) the population of 75 employees.

(b) She could select 10 employees randomly from each department, so that views from each department are represented.

The company operates with a system of Team Leaders. There are 7 Team Leaders in Administration, 6 in Production, 4 in Marketing and 3 in Staffing.

- (c) Find the number of ways in which a working party of 8 Team Leaders can be formed so that there is at least one Team Leader from each of the 4 departments AND there are more Team Leaders from Administration in the working party than from any other single department. [5]

$$\begin{aligned}\text{case [A]: } & 5 \text{ from A, } 1 \text{ from P, } 1 \text{ from M, } 1 \text{ from S} \\ & = {}^7C_5 \times {}^6C_1 \times {}^4C_1 \times {}^3C_1 \\ & = 1512\end{aligned}$$

$$\begin{aligned}\text{case [B]: } & 4 \text{ from A, } 2 \text{ from one other department,} \\ & 1 \text{ each from the other two departments} \\ & = ({}^7C_4 \times {}^6C_2 \times {}^4C_1 \times {}^3C_1) + \\ & \quad ({}^7C_4 \times {}^6C_1 \times {}^4C_2 \times {}^3C_1) + \\ & \quad ({}^7C_4 \times {}^6C_1 \times {}^4C_1 \times {}^3C_2) = 12600\end{aligned}$$

$$\begin{aligned}\text{case [C]: } & 3 \text{ from A, } 1 \text{ from one other department,} \\ & 2 \text{ each from the other two departments} \\ & = ({}^7C_3 \times {}^6C_1 \times {}^4C_2 \times {}^3C_2) + \\ & \quad ({}^7C_3 \times {}^6C_2 \times {}^4C_1 \times {}^3C_2) + \\ & \quad ({}^7C_3 \times {}^6C_2 \times {}^4C_2 \times {}^3C_1) = 19530\end{aligned}$$

$$\text{Total no. of ways} = 1512 + 12600 + 19530 = 33642$$

- 8 (a) The independent random variables X and Y are such that $X \sim N(p, q^2)$ and $Y \sim N(s, t^2)$. Write down an expression for the distribution of the random variable $aX - bY$, where a and b are constants. [2]

$$\begin{aligned} E(aX - bY) &= a E(X) - b E(Y) \\ &= ap - bs \end{aligned}$$

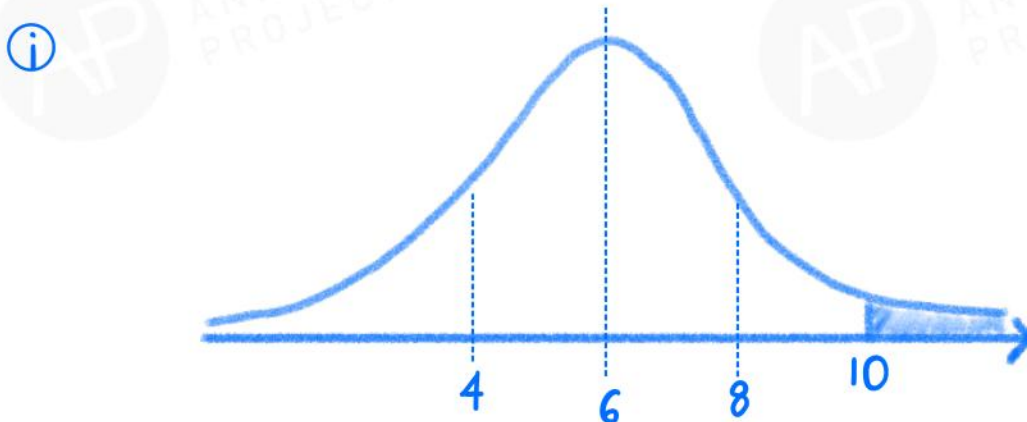
$$\begin{aligned} \text{Var}(aX - bY) &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) \\ &= a^2 q^2 + b^2 t^2 \end{aligned}$$

$$\text{Hence, } aX - bY \sim N(ap - bs, a^2 q^2 + b^2 t^2)$$

- (b) The random variable V is normally distributed with standard deviation 2. The probability that $V > 8$ is equal to the probability that $V < 4$.

(i) Draw a sketch to show the distribution of V , including the main features of the curve. [2]

(ii) On your sketch, shade the area represented by $P(V > 10)$ and state its value. [2]



$$V \sim N(6, 2^2)$$

$$\begin{aligned} \text{② } P(V > 10) &= 0.022750 \\ &= 0.0228 \end{aligned}$$

- (c) The random variable W is such that $W \sim B(8, p)$. The mean of W is 1.2 times the variance of W . Find $P(W < 2)$. [3]

$$E(W) = 1.2 \text{Var}(W)$$

$$8p = 1.2 \times 8p \times (1-p)$$

$$8p = 9.6p(1-p)$$

$$9.6p^2 - 9.6p + 8p = 0$$

$$9.6p^2 - 1.6p = 0$$

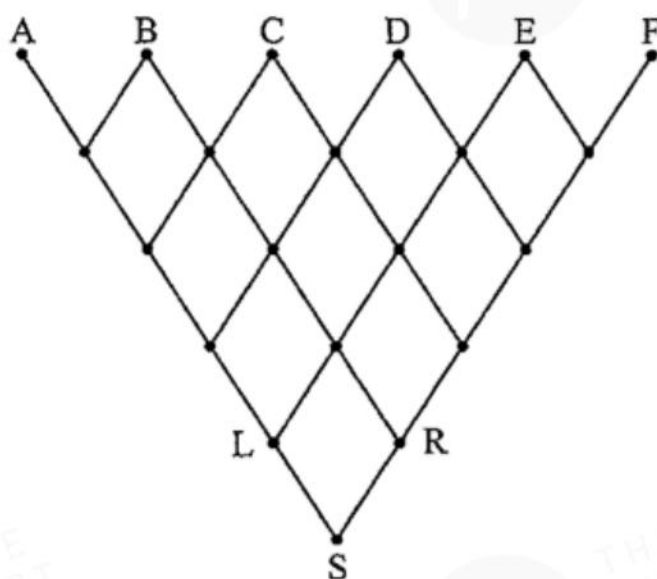
$$1.6p(6p - 1) = 0$$

$$p = 0 \text{ (rej.) or } p = \frac{1}{6}$$

$$\text{By GC: } P(W < 2) = P(W \leq 1) = 0.605$$

9 A game is played using a counter on a board.

- The counter starts from a point S at the bottom of the board and moves upwards (see diagram).
- When leaving S the counter is equally likely to move up to the left to junction L or up to the right to junction R.
- At every junction after S, the counter moves up to the left with probability p or up to the right with probability q , where $p + q = 1$.
- The counter eventually arrives at one of the endpoints A, B, C, D, E or F.



- (a) Show that the probability the counter arrives at B is $2p^3q + \frac{1}{2}p^4$.

[2]

$P(\text{counter arrives at B})$

$$= LRLLL + LLRLL + LLLRL + LLLLR + RLLLL$$

$$= \frac{1}{2}qp^3 + \frac{1}{2}pq p^2 + \frac{1}{2}p^2qp + \frac{1}{2}p^3q + \frac{1}{2}p^4$$

$$= 2p^3q + \frac{1}{2}p^4 \quad (\text{shown})$$

Jon and Kath each play this game, and their counters both arrive at B.

- (b) Find, in terms of p , a simplified fraction for the probability that Jon's and Kath's counters followed exactly the same route.

[4]

$P(\text{Both Jon and Kate followed same route} \mid \text{Both arrive at B})$

$$= \frac{(LRLLL)^2 + (LLRLL)^2 + (LLLRL)^2 + (LLLLR)^2 + (RLLLL)^2}{(2p^3q + \frac{1}{2}p^4)^2}$$

$$= \frac{4(\frac{1}{2}qp^3)^2 + (\frac{1}{2}p^4)^2}{(2p^3q + \frac{1}{2}p^4)^2}$$

$$= \frac{4(\frac{1}{4})(1-p)^2 p^6 + \frac{1}{4}p^8}{[2p^3(1-p) + \frac{1}{2}p^4]^2}$$

$$\begin{aligned}
&= \frac{(1-2p+p^2)p^6 + \frac{1}{4}p^8}{(2p^3 - \frac{3}{2}p^4)^2} \\
&= \frac{p^6 - 2p^7 + \frac{5}{4}p^8}{4p^6 - 6p^7 + \frac{9}{4}p^8} \\
&= \frac{\cancel{\frac{1}{4}p^6}(4-8p+5p^2)}{\cancel{\frac{1}{4}p^6}(16-24p+9p^2)} \\
&= \frac{4-8p+5p^2}{(4-3p)^2}
\end{aligned}$$

The probability that a counter arrives at B is the same as the probability that a counter arrives at C.

(c) Find the value of n

[4]

$P(\text{counter arrives at C})$

$$\begin{aligned}
&= (L R R L L + L R L R L + L R L L R + L L L R R + L L R L R + L L R R L) \\
&\quad + (R R L L L + R L R L L + R L L R L + R L L L R) \\
&= 6(\frac{1}{2}p^2q^3) + 4(\frac{1}{2}p^3q^2) \\
&= 3p^2q^3 + 2p^3q^2
\end{aligned}$$

$$\text{Given that } \cancel{2p^3q} + \frac{1}{2}p^4 = 3p^2q^3 + \cancel{2p^3q}$$

$$\frac{1}{2}p^4 = 3p^2(1-p)^2$$

$$p^4 = 6p^2(1-2p+p^2)$$

$$p^4 = 6p^2 - 12p^3 + 6p^4$$

$$\therefore 5p^4 - 12p^3 + 6p^2 = 0$$

$$p^2(5p^2 - 12p + 6) = 0$$

$$\text{By GC: } p = 0, \quad \underline{0.710} \text{ or } 1.69$$

(rej.) (rej.)

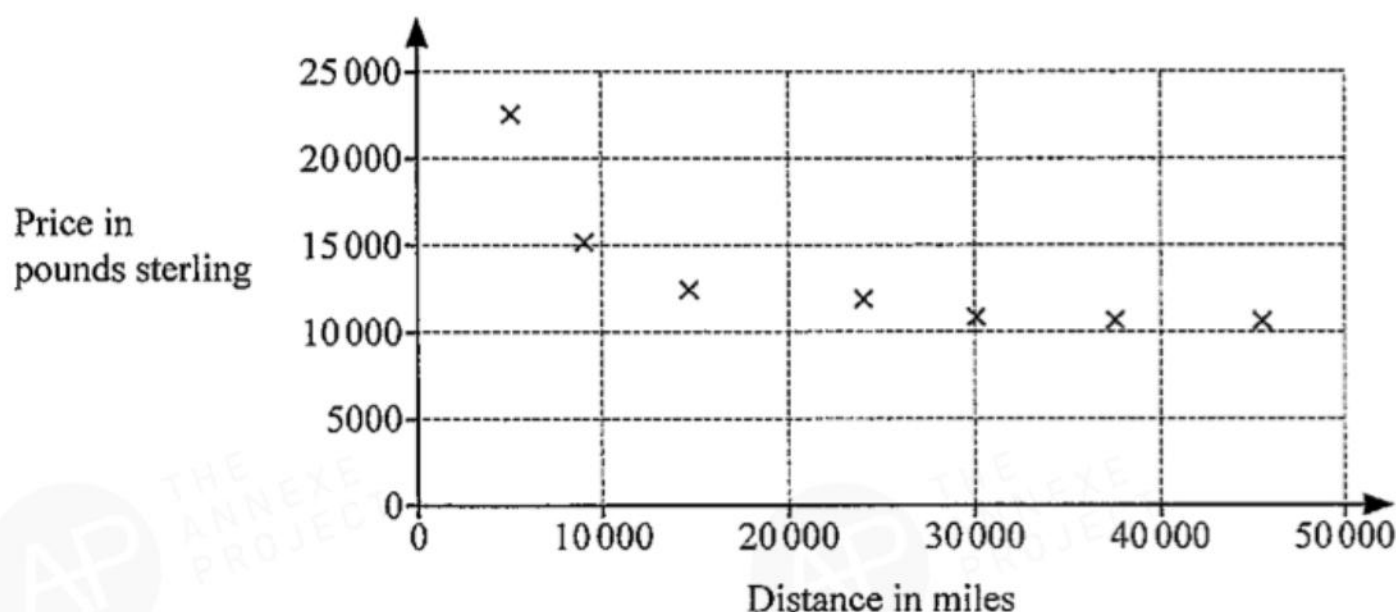


THE ANNEXE PROJECT
EDUCATIONAL CENTRE

- 10 Adriana is investigating how the prices of used cars depend on the distances the cars have been driven. She collects some information from a UK website about 7 cars, all of the same make and model. This is given in the table below; the distance, d , is in miles and the price, p , is in pounds sterling.

d	14 588	8954	24 030	37 551	45 452	30 100	5055
p	12 450	15 200	11 900	10 700	10 600	10 900	22 500

A scatter diagram for the data is shown below.



The product moment correlation coefficient between d and p is -0.78 .

- (a) What conclusion should Adriana reach about the relationship between d and p ? Justify your answer. [2]

- 1). $r = -0.78$ suggests that d and p have a moderate negative linear correlation.
- 2). However, from the scatter plot diagram, p decreases at a decreasing rate as d increases.
Hence, d and p do not have a linear correlation.

- (b) Explain whether the product moment correlation coefficient would differ if Adriana converted the distances to kilometres and the prices to Singapore dollars. [1]

[1 mile = 1.6 kilometres; 1 pound sterling = 1.7 Singapore dollars]

r will not change as it is unaffected by any change of units.

Adriana checks the original information. She realises that the car with $d = 5055$ and $p = 22\,500$ was a special version of the model, with a bigger engine and many extra features.

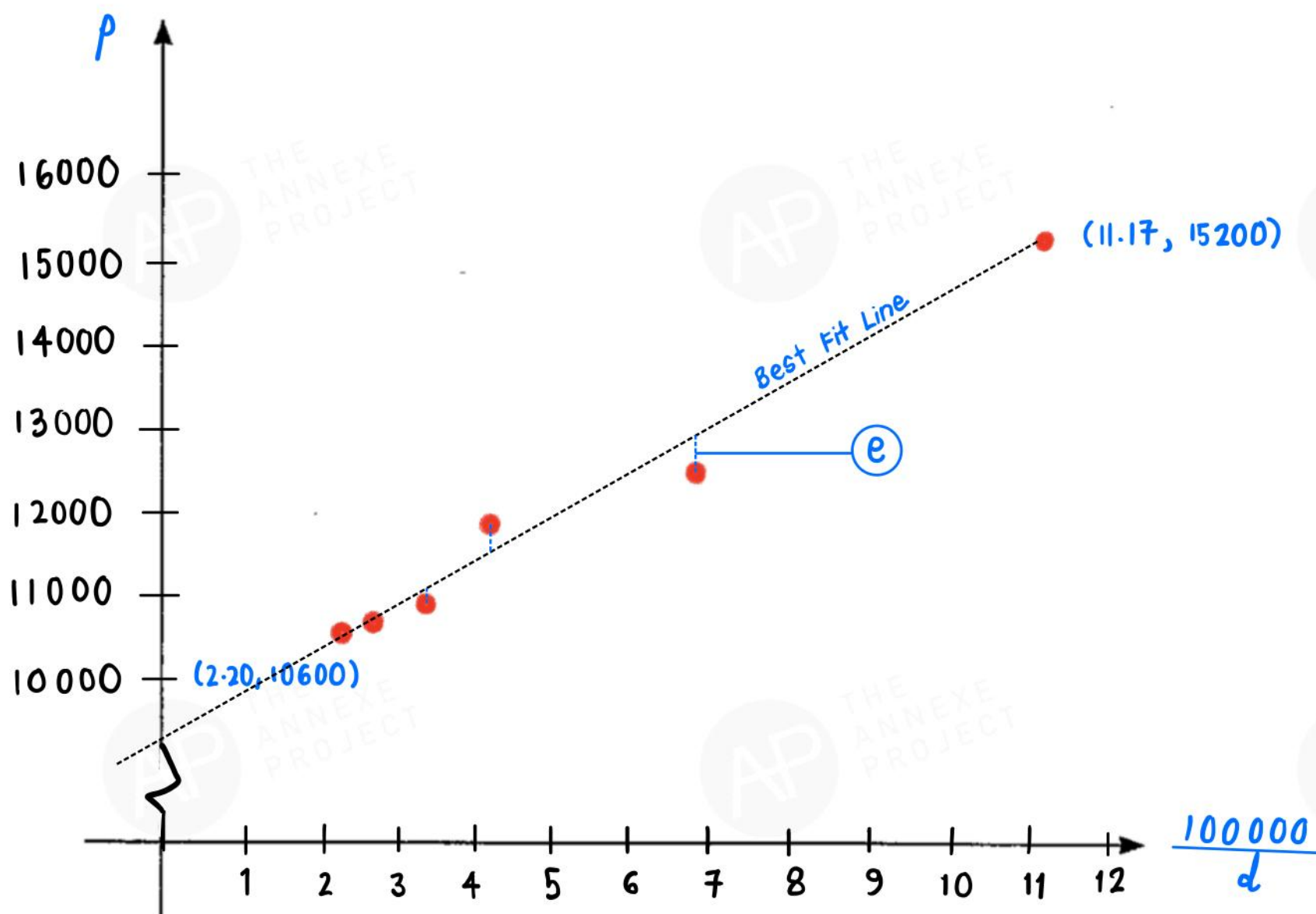
- (c) Explain why the data relating to this special car should not be used in any analysis of the data. [1]

If Adriana is investigating about the cars of the same make and model, then this data would be an outlier and should be removed.

In the rest of this question you should not include the data for this special car.

- (d) Using a reciprocal transformation, sketch a scatter diagram for p against $\frac{100\,000}{d}$. Draw, by eye, a line of best fit on your scatter diagram. [2]

d	14 588	8954	24 030	37 551	45 452	30 100
p	12 450	15 200	11 900	10 700	10 600	10 900
$\frac{100\,000}{d}$	6.85	11.17	4.16	2.66	2.20	3.32



- (e) Show on the scatter diagram in part (d) the distances which are used in drawing the least squares regression line of p on $\frac{100\,000}{d}$. Explain why these distances are squared, and why this is referred to as the 'method of least squares'. [3]

These distances (as seen above) can be positive or negative, they are squared to ensure all values are positive.

In order for a line to be the best fit, the sum of all these distances must be a minimum. Hence, referred as the "method of least squares".

- (f) Use your calculator to find the equation of the least squares regression line of p on $\frac{100\,000}{d}$ and the value of the corresponding product moment correlation coefficient. [3]

By GC: $p = 9399.8 + 505.64 \left(\frac{100\,000}{d} \right)$

$$r = 0.98707$$
$$= 0.987$$

- (g) Use your equation to estimate the price of a car of this make and model that has travelled 5055 miles. Explain whether you would expect this estimate to be reliable. [2]

When $d = 5055$,

$$p = 9399.8 + 505.64 \left(\frac{100\,000}{5055} \right)$$
$$= 19402.57$$

This estimation is unreliable because extrapolation was practised. Data collected for the above regression line was between 8954 miles to 45452 miles.

THE ANNEXE PROJECT EDUCATIONAL CENTRE



THE ANNEXE PROJECT
EDUCATIONAL CENTRE

- 11 Zhou and Tan are both keen amateur swimmers, and several times each week they compete against each other to see who can swim 100 metres freestyle in the shortest time. Over a long period, Zhou's times, in seconds, have followed the distribution $N(80, 2^2)$ and Tan's the distribution $N(79, 3^2)$. It is assumed that these two distributions are independent.

(a) Find the probability that Zhou wins a randomly chosen race.

[3]

Let Zhou's time be $Z \sim N(80, 2^2)$

Let Tan's time be $T \sim N(79, 3^2)$

$$Z - T \sim N(1, 13)$$

$$\begin{aligned} P(Z < T) &= P(Z - T < 0) \\ &= 0.39076 \\ &= 0.391 \end{aligned}$$

Tan moves abroad for 2 years so the regular races are suspended. Zhou starts an exercise regime that he believes will reduce his times. After he has been exercising for a year, he records his time, x seconds, for a random sample of swims. These times are summarised below.

$$n = 30 \qquad \Sigma x = 2376.3 \qquad \Sigma x^2 = 188653.7$$

(b) Calculate unbiased estimates of the population mean and variance for Zhou's times after his exercise regime.

[2]

$$\bar{x} = \frac{\Sigma x}{n} = \frac{2376.3}{30} = 79.21$$

$$\begin{aligned} s^2 &= \frac{1}{n-1} \left[\Sigma x^2 - \frac{(\Sigma x)^2}{n} \right] = \frac{1}{29} \left[188653.7 - \frac{2376.3^2}{30} \right] \\ &= 14.723 \end{aligned}$$

(c) State hypotheses that can be used to test if Zhou's times have reduced, defining any symbols you use. Work out the test statistic in this case, and use it to carry out the test at the 5% level of significance, giving your conclusion in the context of the question.

[5]

Step 1:

Let X be the random variable denoting Zhou's time (in seconds) to swim 100m freestyle after the exercise regime.

Since $n=30$ is large, by CLT, $\bar{X} \sim N(80, \frac{14.723}{30})$ approximately.

Step 2:

To test $H_0: \mu = 80$ against

$H_1: \mu < 80$ at 5% level of significance

(H_0 : Null Hypothesis, H_1 : Alternate Hypothesis, μ : population mean)

Step 3:

Test Statistic:
$$Z = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \sim N(0, 1)$$

$$= \frac{79.21 - 80}{\sqrt{\frac{14.723}{30}}} = -1.1277$$

By GC: $p\text{-value} = 0.12973$

Step 4:

Since $p > 0.05$, H_0 is not rejected.

There is insufficient evidence at 5% level of significance to conclude that Zhou's times have reduced.

While he is away, Tan switches to a high protein diet and wonders whether this will affect the times he takes for his swims. After a year Tan decides to test, at the 2.5% level of significance, whether his mean time has changed. He records a random sample of 6 times and calculates the mean of these times.

(d) Explain whether Tan should use a 1-tail or a 2-tail test. [1]

He should a 2-tail test as Tan did not state if he would like to check if his times have increased or reduced.

(e) Tan makes two assumptions and is then able to carry out his test. State these two assumptions. [2]

1. He assumes that the times he takes for his swim follows a Normal Distribution.
2. He assumes that each time he took for a swim is independent to one another.



THE ANNEXE PROJECT
EDUCATIONAL CENTRE

ESTD 2009